Correlation-Savvy Sellers

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Abstract

A multi-product monopolist sells products sequentially to a buyer who privately learns his valuations. Using big data, the monopolist perfectly learns the intertemporal correlations of the buyer’s valuations. Despite her perfect learning, perfect price discrimination is generally unattainable—even under full commitment. Due to an informational externality between the consumer’s initial valuation and the correlation structure, the seller has to concede information rents also for the buyer’s consumption in later periods. Upward distortions are a robust feature of the profit maximizing outcome; the presence of distortions are non-monotonic in the seller’s data-mining abilities.

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1 Introduction

In recent decades, the tracking of consumers has become standard practice in both online and offline consumer markets. In online markets, retailers not only track their customers as soon as they log into their personal accounts, they also track them through the use of cookies when they are not logged in. Brick and mortar stores track their consumers by linking scanner data to credit card payments and by setting up loyalty or membership programs, which, in terms of tracking, play a similar role to the consumer’s online account at an online shop. All this tracking yields massive amounts of data—big data, through which retailers sift continuously with the help of data scientists.

While data science is hailed for revolutionizing consumer markets by the use of highly sophisticated computing techniques, boiled down to its essence, it tries to identify robust correlations that are potentially valuable to retailers. One of the first and by now classical example is the detection by a data scientist in 1992 of an unexpected positive correlation between the sales of beer and diapers, prompting the retailer to group together the two products, and thereby raising sales[1] While novel at the time, it is now standard that based on big data, online platforms such as Amazon and Netflix make personalized suggestions for buying products and viewing movies[2] It has also become more and more prevalent to use this information for sending out personalized vouchers, thereby allowing retailers to price discriminate.

Yet, even though the high investments of retailers into data science techniques signify their practical importance, our basic economic understanding of sellers learning such correlations is still limited. Indeed, most of our insights about the impact of big data are based on models that consider sellers who learn directly the private information of a particular consumer. It is however unclear whether the insights of such models also apply to sellers who learn only about the correlations of consumers’ tastes rather than a consumer’s specific traits.

To study this issue, the paper develops an explicit model of a “correlation-savvy seller”, who learns only about the correlation of a buyer’s valuations but does so perfectly. Its analysis reveals that learning about correlation differs qualitatively from learning about a consumer’s type directly. In particular, learning about correlation leads to an informational externality between the consumer’s initial valuation and the correlation structure. This externality is novel and is not present when a seller learns about a consumer’s type directly. It is a main driver of information rents and, thereby, generates the following two policy relevant insights. First, a correlation-savvy seller is generally unable to attain perfect price discrimination even if she learns the correlation perfectly. Second, upward distortions in quantities are a robust feature when learning about correlations.

We obtain these insights from considering a two period-version of the dynamic mechanism design problem of Battaglini (2005) and extending it by including a stage in which the seller perfectly learns about the transitions of the buyer’s valuations. The resulting framework is tractable; it enables a complete characterization of the seller’s profit-maximizing mechanism for all possible parameter con-


[2]This indicates that big data enable sellers to get to know the tastes of their customers better than the consumers themselves. See Hariri (2018) for an elaborate discussion of the fact that data science allows to obtain better knowledge about individuals than the individuals have themselves.
stellations. This characterization shows that distortions crucially depend on the expected degree of persistence. Despite our assumption that the monopolist learns the correlation structure perfectly, the outcome of perfect price discrimination—implementing efficient quantities without conceeding any information rents—is only attainable when the persistence probabilities are moderate in that neither negative nor positive persistence are too likely. By contrast, if the likelihood of a positive or a negative persistence is relatively high, perfect price discrimination is unattainable.

In either of these more polarized cases, the profit-maximizing outcome displays downward as well as upward distortions. These upward distortions point to the novel economic force that is not present in models of monopolistic screening: The informational dependency that the buyer's ex ante private information is complementary to the seller's private information. Due this complementarity, the buyer's ex ante private information determines not only the information rents with respect to the consumption associated with this private information, but also the information rents with respect to the other good's consumption. Models of monopolistic screening only exhibit the former, internal role of private information but not the latter, external role. Indeed, it is this external role that is responsible for the two insights that, first, the perfect price discrimination is not attainable for higher levels of correlation, and, second, that upward monopolistic distortions are a robust feature whenever perfect price discrimination is not attainable.

In addition to the informational complementarity effect, we identify a second informational dependency that also arises naturally in a framework of learning about correlations: Given a truthful revelation of the buyer's ex ante information, the seller's private information is correlated with the buyer's ex post private information. In the spirit of Crémer and McLean (1985), this informational dependency implies that the seller has to pay information rents only for inducing the buyer to reveal his ex ante private information. This feature is crucial for yielding a tractable model, because it implies that direct mechanisms are optimal despite the fact that the seller's commitment is limited (as she cannot precommit to report her private information about the correlation honestly). Moreover, it implies that the model is as if the observed persistence is verifiable and the seller can directly condition her contract in this information. For this reason, a model in which the buyer's persistence is learned publicly yields identical results. There is therefore no role for policies regulating the observability of correlations.

Relating our findings to the literature, the current paper belongs to the strand of literature that considers optimal pricing policies of monopolists, who are able to learn about the consumers' purchase history and/or their personal tastes. Indeed, Battaglini (2005) shows that when the transition matrix is stochastic and the seller does not learn about correlations, optimal schedules only exhibit downward distortions. In contrast, Acquisti and Varian (2005) study dynamic pricing with consumers who have fully persistent types so that the transition matrix is deterministic, leaving no room for learning about correlations. The authors show that with sophisticated (i.e., fully strategic) consumers, the optimal dynamic schedule is simply a repetition of the static outcome that exhibit the usual downward distortions.

Models of behavior-based price discrimination (see Fudenberg and Villas-Boas, 2006) typically consider a different type of learning: The ability of firms to learn about whether they are facing old or

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For this reason, Laffont and Tirole (1993) who study fully persistent adverse selection models in the context of regulation, say that these full persistence models exhibit “false dynamics”.

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new consumers. Consequently, these models study intertemporal price-discrimination; the ability to charge consumers of a different “buying age” different prices. That intertemporal price discrimination is optimal, already follows from Battaglini (2005) who obtains that, with partially persistent types, the optimal dynamic pricing schedule is time-dependent.

By focusing on correlation, learning in our model differs from studies who consider monopolistic sellers that learn about a consumer’s private information directly and use this information to price discriminate (e.g., Conitzer et al. 2012, Bergemann et al. 2015, De Cornière and De Nijs 2016, Ali et al. 2019, Bonatti and Cisternas 2020, De Cornière and Taylor 2020). In contrast to the current paper, perfect learning in this literature allows a monopolist to attain perfect price discrimination. Studying a different type of learning, our analysis complements this literature.

Contrasting complementarities between the seller’s and the buyer’s private information as studied here, a recent literature studies data externalities (e.g., Choi et al. 2019, Acemoglu et al. 2019, Bergemann et al. 2019, Ichihashi 2021). Because they reflect informational interdependencies between the consumers themselves, these externalities are orthogonal to the information externalities that we study.

For a more fundamental discussion of modeling informational externalities we refer to Börgers et al. (2013).

2 Model

Consider a seller (she), who provides a quantity of some good \( q \) to a buyer (he) and subsequently some quantity of some good \( Q \). The transaction involves an overall transfer \( T \in \mathbb{R} \) from the buyer to the seller. Hence, the economic allocation is a triple \((T, q, Q) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \). We are agnostic about the physical relationship between the goods \( q \) and \( Q \). That is, they can represent the same good so that the model is one of repeated purchases, or \( Q \) may pertain to goods that are physically unrelated to good \( q \), such as beer and diapers.

Measuring the importance of good \( Q \) relative to good \( q \) by a parameter \( \omega \geq 0 \), the seller’s profit and the buyer’s utility associated with an allocation \((T, q, Q) \) are, respectively,

\[
\Pi(T, q, Q) = T - c(q) - \omega C(Q) \quad \text{and} \quad U(T, q, Q | \theta, \Theta) = \theta q + \omega \Theta Q - T, 
\]

where \( c(.) \) and \( C(.) \) represent the seller’s cost functions, and \((\theta, \Theta) \) the agent’s marginal valuation for quantities \( q \) and \( Q \), respectively. We assume that the cost functions are twice differentiable, increasing, convex, and exhibit \( c(0) = c'(0) = C(0) = C'(0) = 0 \) and \( \lim_{q \to \infty} c(q) = \lim_{q \to \infty} c'(q) = \lim_{Q \to \infty} C(Q) = \lim_{Q \to \infty} C'(Q) = \infty \). Concerning the valuation \((\theta, \Theta) \), we assume that they are binary; they can either be high, \( h \), or low, \( l \), with \( \Delta \equiv h - l > 0 \).

Efficient quantities equalize marginal costs to marginal utility, and, hence, the efficient quantities \((q^*_h, q^*_l, Q^*_h, Q^*_l) \) satisfy

\[
c'(q^*_h) = h; \quad c'(q^*_l) = l; \quad C'(Q^*_h) = h; \quad C'(Q^*_l) = l.
\]

Our assumptions on the cost functions together with \( \Delta > 0 \) imply \( q^*_h > q^*_l > 0 \) and \( Q^*_h > Q^*_l > 0 \).

Reflecting the sequential structure, we refer to the valuation \( \theta \) as the buyer’s ex ante type, and the valuation \( \Theta \) as the buyer’s ex post type. We can fully describe the correlation structure of \( \theta \) and \( \Theta \).
by specifying the conditional probability $P(\Theta|\theta)$ as displayed in Table 1. In our model with binary types, we can alternatively represent the correlation structure by a pair $(\pi_h, \pi_l)$, describing the (right) stochastic matrix

$$M = \begin{pmatrix}
\pi_h & 1 - \pi_h \\
1 - \pi_l & \pi_l
\end{pmatrix}.$$ 

If there is perfect knowledge about the transition, the transition matrix is deterministic and takes on one of the following four forms

$$M_+ \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad M_- \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad M_h \equiv \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}; \quad M_l \equiv \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

where $M_+$ describes the case of positive persistent types, $M_-$ the case of negative persistence types, and the latter two matrices, $M_h$ and $M_l$, the absorbing cases where, regardless of their ex ante types $\theta$, agents have ex post type $\Theta = h$ and $\Theta = l$ respectively.

It is well known (see Davis, 1961) that we can decompose any stochastic matrix as a convex combination of its deterministic matrices. That is, in our binary context there are weights $\lambda_+, \lambda_-, \lambda_h, \lambda_l \geq 0$ with $\lambda_+ + \lambda_- + \lambda_h + \lambda_l = 1$ such that

$$M = \lambda_+ M_+ + \lambda_- M_- + \lambda_h M_h + \lambda_l M_l.$$ 

Interpreting the weights as probabilities yields the interpretation that any non-degenerate stochastic matrix represents a situation in which the true deterministic transition matrix is uncertain. Motivated by this interpretation, we say that learning about the correlation structure perfectly means finding out which of the deterministic matrices represents the true state.

When types are binary, there are in general 4 possible states. Since only 2 states suffice for demonstrating our main results, we simplify and study economic settings in which the agent’s persistence is either positive ($p = +$) or negative ($p = -$). Denoting the probability of positive persistent types by $\pi$, the stochastic transition matrix corresponds to the (symmetric) stochastic matrix

$$M = \pi M_+ + (1 - \pi)M_- = \begin{pmatrix} \pi & 1 - \pi \\ 1 - \pi & \pi \end{pmatrix}.$$ 

Assuming that the buyer’s ex ante type $\theta$ is $h$ with probability $P(\theta = h) = \nu$, while it is $l$ with
probability $P(\theta = l) = 1 - \nu$, the joint probability distribution of $(\theta, \Theta)$ is

$$P\{(\theta, \Theta) = (h, h)\} = \nu \pi; \quad P\{(\theta, \Theta) = (h, l)\} = \nu(1 - \pi);$$

$$P\{(\theta, \Theta) = (l, l)\} = (1 - \nu) \pi; \quad P\{(\theta, \Theta) = (l, h)\} = (1 - \nu)(1 - \pi).$$

As motivated in the introduction, we consider an environment where big data enables the seller to learn the true underlying deterministic transition matrix perfectly. This is equivalent to learning the buyer’s persistence $p$. In particular, we start with considering the following timing:

0. Buyer privately learns $\theta$;
1. Seller offers a contract determining the terms of trade $(T, q, Q)$;
2. Buyer decides whether to reject or accept the contract;
3. Seller learns $p$;
4. Consumption of quantity $q$;
5. Buyer privately learns $\Theta$;
6. Consumption of quantity $Q$.

This timing presumes that the seller learns $p$ only after offering the contract but before the buyer consumes $q$. Starting with this timing does not only yield the most tractable model, it also allows us to show that the exact timing of when the seller learns $p$ does not matter. In particular, the equilibrium outcome that we obtain with this timing stays an equilibrium outcome in a model in which the seller learns $p$ before offering the contract, or in a model in which the seller learns $p$ only after the consumption of quantity $q$. Under the former alternative, we obtain a signalling game with additional but less profitable equilibrium outcomes, while under the latter alternative, we obtain the same equilibrium outcome, because the optimal mechanism that we identify is also implementable if the seller learns $p$ only after the consumption of $q$. We discuss the robustness of our results to these alternative timings in our conclusion.

Note that without stage 3, the model reduces to a two-period version of the dynamic mechanism design problem of Battaglini (2005). For this case, we know that the profit-maximizing contract corresponds to an incentive compatible direct mechanism that induces the buyer to first report truthfully his $\theta$, and, after learning $\Theta$, subsequently to report this value truthfully. Consequently, we can express the optimal contract as a triple $(\tilde{T}_{\tilde{\theta} \tilde{\theta}}, \tilde{q}_{\tilde{\theta}}, \tilde{Q}_{\tilde{\theta} \tilde{\theta}})$ that conditions on a report $\tilde{\theta}$ about $\theta$ and a report $\tilde{\Theta}$ about $\Theta$, and satisfies incentive constraints for reporting these values truthfully. With the addition of stage 3, where the seller learns the transition matrix perfectly, the question arises whether an appropriate extension of such an incentive compatible direct mechanism also represents a profit-maximizing contract. We explicitly address this question in Section 4.

Moreover, note that while a model in which the buyer learns $\Theta$ is equivalent to one in which he learns $p$, a model in which the seller learns $p$ and one in which she learns $\Theta$ would not be equivalent. This is so, because in order to learn $\Theta$, the information about $p$ is insufficient as it requires also knowing

\footnote{Hence, the persistence of types, $p$, and the ex ante type, $\theta$, are stochastically independent so that the seller cannot exploit the correlation between $p$ and $\theta$ as in Riordan and Sappington (1988).}
This observation already indicates that \( p \) and \( \theta \) reflect an informational complementarity. As shown in the analysis, it is this complementarity that makes perfect price discrimination unattainable.

## 3 Benchmarks

### 3.1 Perfect Price Discrimination Benchmark

First consider the outcome under perfect information, where the seller directly observes the agent’s overall type \((\theta, \Theta)\). In this case, the seller can price discriminate perfectly and extract the whole surplus by offering type \((\theta, \Theta)\) the type-specific efficient quantities \(q^*_\theta\) and \(Q^*_\Theta\) for an overall transfer of \(T = \theta q^*_\theta + \omega \Theta Q^*_\Theta\). Using this scheme, the seller implements the perfect price discrimination outcome

\[
q_{l+} = q_{l-} = q^*_l < q_{h+} = q_{h-} = q^*_h; Q_{l+l} = Q_{h-l} = Q^*_l < Q_{h+h} = Q^*_h; U_h = U_l = 0. 
\]

### 3.2 Static Benchmark

As a second benchmark, it is useful to consider a pure static version of the model in which learning about the correlation structure does not matter. We obtain such a model for the case \(\omega = 0\), where the model boils down to a static, single-good monopolistic screening problem in the tradition of Mussa and Rosen (1978) with binary types. In this classical setup, it is optimal for the seller to offer an incentive compatible menu \{(\(q_l, p_l\), \(q_h, p_h\))\} that maximizes her expected profits

\[
\Pi = \nu(p_h - c(q_h)) + (1 - \nu)(p_l - c(q_l))
\]

subject to the incentive compatibility constraints

\[
hq_h - p_h \geq hq_l - p_l \text{ and } lq_l - p_j \geq lq_h - p_h;
\]

and the individual rationality conditions

\[
hq_h - p_h \geq 0 \text{ and } lq_l - p_l \geq 0.
\]

As is well known, the seller’s first best outcome of selling the efficient quantity \(q^*_\Theta\) to either type \(\Theta\) at a price \(\theta q^*_\Theta\), is not attainable as it violates the incentive constraint of the efficient type \(h\). Indeed, at the optimum, the incentive constraint of the efficient and the individual rationality of the inefficient type constrain the seller and induces her to distort the quantity \(q_l\) downwards. In particular, it leads the seller to offer type \(l\) a quantity \(q^*_l < q^*_h\) such that\(^5\)

\[
c'(q^*_l) = (l - \varphi \Delta)^+,
\]

where \(\varphi \equiv \nu/(1 - \nu)\) is the relative likelihood of an efficient type. The following proposition summarizes the outcome in this benchmark.

\(^5\)To deal with corner solutions, let \((x)^+\) denote the positive part of a number \(x\), i.e. \((x)^+ \equiv \max\{0, x\}\).
Proposition 1 Suppose $\omega = 0$. Then it is optimal for the seller to offer the incentive compatible menu $\{(q_l, p_l), (q_h, p_h)\}$ with the efficient quantity $q_h = q^*_h$ for type $h$ and a downward distorted quantity $q_l = q^*_l < q^*_h$ for type $l$, leaving a positive rent to type $h$ and no rents to type $l$.

4 Direct Mechanisms and Incentive Compatibility

Following the usual approach in mechanism design, it is natural to first consider the use of incentive compatible direct mechanisms. In a dynamic framework, such mechanisms induce the players to report their private information confidentially and truthfully as soon as they obtain it, and condition the allocation $(T, q, Q)$ only on these confidential reports (see Myerson, 1986). For our context this means that the buyer reports his valuation $\theta$ after accepting the mechanism (but before learning $\Theta$); the seller’s reports $p$ after learning it (but without knowing the buyer’s report); and the buyer reports $\Theta$ after learning it (but without knowing the seller’s report). That is, a direct mechanism is a triple

$$\gamma = (T_{\hat{\theta} \hat{p} \hat{\theta}}, q_{\hat{\theta} \hat{p} \hat{\theta}}, Q_{\hat{\theta} \hat{p} \hat{\theta}}),$$

that conditions the economic allocation $(T, q, Q)$ on the buyer’s reports $\hat{\theta}$ and $\hat{\Theta}$ about the valuations $\theta$ and $\Theta$, respectively, and on the seller’s report $\hat{p} \in \{0, 1\}$ about the persistence $p$. Note that since the buyer learns $\Theta$ after consuming the quantity $q$, only the overall transfer $T$ and the quantity $Q$ can condition on the report $\hat{\Theta}$.

Given our specific context, we however have to address two potential problems concerning the applicability of such direct mechanisms. First, note that we have here a setting with limited commitment, since the seller as the mechanism designer is to report her private information to the mechanism itself, signifying an action to which the seller cannot pre-commit. However, because we model a direct mechanism as one for which the buyer’s report about $\theta$ is only revealed to the seller after she sends her report $\hat{p}$, this limited commitment is innocuous.

A second potential problem is whether the required confidentiality of the reporting is actually feasible, i.e., whether such confidentiality stands in conflict with the actual execution of the direct mechanism. This seems indeed the case, because if the direct mechanism $\gamma$ conditions the quantity $q$ non-trivially on $\hat{p}$, the buyer can deduce the seller’s report $\hat{p}$ from her consumption of $q$ before reporting $\Theta$. This however undermines the presumed confidentiality of $\hat{p}$. Our solution to this problem is to side-step this issue completely (i.e., implicitly assuming that such deductions do not take place), and show that the profit-maximizing direct mechanism optimally does not condition the quantity $q$ on the report $\hat{p}$ so that deducing $\hat{p}$ from $q$ is indeed not possible.

Summarizing, a direct mechanism $\gamma$ induces a game in which the buyer first learns his ex ante type $\theta$ about which he sends a report $\hat{\theta}$, subsequently the seller learns $p$ about which he sends a report $\hat{p}$ without observing the buyer’s report $\hat{\theta}$. Finally, the buyer learns his ex post type $\Theta$ about which he sends a report $\hat{\Theta}$, while being ignorant of the seller’s report $\hat{p}$. The outcome of the game is the allocation $(T_{\hat{\theta} \hat{p} \hat{\theta}}, q_{\hat{\theta} \hat{p} \hat{\theta}}, Q_{\hat{\theta} \hat{p} \hat{\theta}})$, inducing the respective payoffs of the seller and buyer.

We next address the issue of incentive compatibility: which direct mechanisms $\gamma$ induce a game as described above in which truthful reporting is an equilibrium? Our first observation is that, by exploiting
the correlation between \( p \) and \( \Theta \), the seller can costlessly ensure that she reports honestly in the sense that it allows her to behave as if she could pre-commit to a truthful reporting strategy. To show this formally, note that if both the seller and the buyer report truthfully, the following sequences of reports \((\tilde{\theta}, \tilde{p}, \tilde{\Theta})\) never occur:

\[
\varnothing \equiv \{(l, -, l), (l, +, h), (h, -, h), (h, +, l)\}.
\]

This feature enables us to induce the seller to report \( p \) honestly without paying any rents. To see this, note that if the seller expects the buyer to report truthfully, then she has an incentive to report \( p = 1 \) truthfully if

\[
\nu \Pi(T_{h+l}, q_{h+l}, Q_{h+l}) + (1 - \nu) \Pi(T_{l-h}, q_{l-h}, Q_{l-h}) \geq \nu \Pi(T_{h-l}, q_{h-l}, Q_{h-l}) + (1 - \nu) \Pi(T_{l-h}, q_{l-h}, Q_{l-h}).
\]

Hence, by setting \( Q_{h-l} \) and \( Q_{l-h} \) large enough, the seller has an incentive to report truthfully. Since the reporting triples \((h, -, h)\) and \((l, -, l)\) do not occur on the equilibrium path, the equilibrium payoffs do not depend on them. Hence, ensuring incentive compatibility by setting them large does not come at any economic costs. Likewise, the seller has an incentive to report \( p = 0 \) truthfully if the out-of-equilibrium quantities \( Q_{h+l} \) and \( Q_{l+h} \) are large enough.

Similarly by setting the out-of-equilibrium transfers \( T_{l-h}, T_{h-l}, T_{l-h}, T_{h-l} \) large enough, we can ensure that, given a truthful report \( \theta \), the buyer also reports his private information \( \Theta \) truthfully. Hence, by punishing both the buyer and the seller for an out-of-equilibrium reporting triple in \( \varnothing \), we can elicit \( p \) and \( \Theta \) costlessly.

This means that the only relevant incentive constraint is the one that induces the agent to report his ex ante type \( \theta \) truthfully. That is, the constraint prevents the buyer from misreporting \( \theta \) and coordinate this misreporting with his report \( \tilde{\Theta} \).

To derive this incentive constraint formally, note that the buyer’s action is a report \( \tilde{\theta} \in \{h, l\} \) about \( \theta \) and a report \( \tilde{\Theta} \in \{h, l\} \) about \( \Theta \). Since from observing \((\theta, \Theta)\) the buyer can perfectly deduce the seller’s observation of \( p \), the buyer can condition his report \( \tilde{\Theta} \) upon the seller’s observation of \( p \). This means that the buyer effectively decides about a triple \( s = (s_1, s_2, s_3) \in S \equiv \{l, h\} \times \{l, h\} \times \{l, h\} \), where \( s_1 \) represents the report \( \tilde{\theta} \), \( s_2 \) represents the report \( \tilde{\Theta} \) given there is negative persistence \( p = - \), and \( s_3 \) represents the report \( \tilde{\Theta} \) given there is positive persistence \( p = + \). Given the buyer expects the seller to report her observed \( p \) truthfully, the payoff associated with \( s = (s_1, s_2, s_3) \) of a buyer with private information \( \theta \) is

\[
U(s_1, s_2, s_3|\theta) = \pi[q_{s_1} + \theta + \omega Q_{s_1} + \theta - T_{s_1 + s_2}] + (1 - \pi)[q_{s_3} - \theta + \omega Q_{s_3} - \theta - T_{s_1 + s_3}],
\]

where \( \omega \) is the singleton of the set \( \{l, h\}\)\{\{\theta\}\}.

For an ex ante type \( \theta = h \), honest reporting means to pick the triple \((s_1, s_2, s_3) = (h, l, h)\) rather than

\(\text{As formally shown in the proof of Lemma 1, the result that we can, at the same time, punish both the seller and the buyer for these out-of-equilibrium reports holds because the aggregate surplus } S(Q) = \Theta Q - C(Q) \text{ goes to minus infinity as } Q \text{ becomes large. Note that the confidentiality of the seller’s report } \hat{p} \text{ plays an important role here, because it safeguards the buyer from the “extortion” that by misreporting } p, \text{ the seller can force the buyer to report a certain } \Theta \text{ in order to avoid the subsequent punishment. Moreover note, that while the perfect correlation between } p \text{ and } \Theta \text{ simplifies the argument, it is not crucial; with an imperfect correlation, incentive compatibility obtains by involving a third party (e.g., an internet platform).}
\( (l, h, l) \). Hence, truth-telling requires \( U(h, l, h|h) \geq U(l, h, l|h) \), yielding the incentive constraint:

\[
\pi[q_h + \omega Q_{h+h}h - T_{h+h}] + (1 - \pi)[q_h + \omega Q_{h-l}l - T_{h-l}] \geq 0
\]

\[\quad \pi[q_l + \omega Q_{l+l}l - T_{l+l}] + (1 - \pi)[q_l + \omega Q_{l-h}l - T_{l-h}]. \tag{4}\]

For an ex ante type \( \theta = l \), honest reporting means to pick the triple \((s_1,s_2,s_3) = (l, h, l)\) rather than \((h, l, h)\). Hence, truth-telling requires \( U(l, h, l|l) \geq U(h, l, h|l) \), yielding the incentive constraint:

\[
\pi[q_h + \omega Q_{h+h}h - T_{h+h}] + (1 - \pi)[q_h + \omega Q_{h-l}l - T_{h-l}] \geq 0
\]

\[\quad \pi[q_l + \omega Q_{l+l}l - T_{l+l}] + (1 - \pi)[q_l + \omega Q_{l-h}l - T_{l-h}]. \tag{5}\]

Concerning the buyer’s acceptance of the mechanism, note that a direct mechanism which satisfies the incentives constraints \( (IC_l) \) yields a buyer of ex ante type \( \theta = h \) at least his outside option of zero if and only if

\[
\pi[q_h + \omega Q_{h+h}h - T_{h+h}] + (1 - \pi)[q_h + \omega Q_{h-l}l - T_{h-l}] \geq 0 \tag{6}\]

Likewise, a direct mechanism that satisfies the incentives constraints \( (IC_h) \) yields a buyer of ex ante type \( \theta = l \) at least his outside option of zero if and only if

\[
\pi[q_l + \omega Q_{l+l}l - T_{l+l}] + (1 - \pi)[q_l + \omega Q_{l-h}l - T_{l-h}] \geq 0 \tag{7}\]

We say a direct mechanism \( \gamma = (T_\theta\hat{\theta}, q_\theta\hat{\theta}, Q_\theta\hat{\theta}) \) is feasible if it satisfies the incentive compatible constraints \((4)\) and \((5)\) and the individual rational constraints \((6)\) and \((7)\). The previous reason leads to the following lemma, which represents a revelation principle for our setup:

**Lemma 1** Suppose \( \omega > 0 \). Then there is no loss in focusing on direct mechanisms \( \gamma = (q_\theta\hat{\theta}, Q_\theta\hat{\theta}, T_\theta\hat{\theta}) \) that are feasible, provided that the optimal direct mechanism \( \gamma^* \) exhibits \( q_{l\theta}^* = q_{l\theta}^ \) and \( q_{h\theta}^* = q_{h\theta}^ \).

The lemma motivates to consider the problem of maximizing the seller’s payoff

\[
\Pi = v(\pi[T_{h+h} - c(q_{h+h}) - \omega C(Q_{h+h})] + (1 - \pi)[T_{h-l} - c(q_{h-l}) - \omega C(Q_{h-l})])
\]

\[\quad + (1 - v)(\pi[T_{l+l} - c(q_{l+l}) - \omega C(Q_{l+l})] + (1 - \pi)[T_{l-h} - c(q_{l-h}) - \omega C(Q_{l-h})]) \]

with respect to \( \gamma \) and subject to the constraints \((4)\), \((5)\), \((6)\), and \((7)\). If its solution \( \gamma^* \) does not condition \( q \) on \( \hat{\theta} \), then the solution represents the seller’s profit-maximizing mechanism.

Rather than working with the quantities and transfers, it is often more convenient to reformulate the problem in terms of quantities and information rents. Hence, define the information rent of ex ante type \( \theta = h \) as

\[
U_h \equiv \pi[q_h + \omega Q_{h+h}h - T_{h+h}] + (1 - \pi)[q_h + \omega Q_{h-l}l - T_{h-l}];
\]

and the information rent of ex ante type \( \theta = l \) as

\[
U_l \equiv \pi[q_l + \omega Q_{l+l}l - T_{l+l}] + (1 - \pi)[q_l + \omega Q_{l-h}l - T_{l-h}].
\]
It then follows that the incentive compatibility constraints (4) and (5) rewrite as

\[ U_h \geq U_l + \pi[q_{l+} + \omega Q_{l+h}]\Delta + (1 - \pi)(q_{l-} - \omega Q_{l-h})\Delta; \]  
(8)

\[ U_l \geq U_h - \pi[q_{h+} + \omega Q_{h+h}]\Delta - (1 - \pi)(q_{h-} - \omega Q_{h-l})\Delta. \]  
(9)

while the individual rationality constraints (6) and (7) simplify to

\[ U_h \geq 0; \]  
(10)

\[ U_l \geq 0. \]  
(11)

Rewriting the seller’s ex ante expected profit of a feasible direct mechanism \( \Pi \) in terms of quantities and rents, we obtain

\[
\Pi = \nu \{ \pi[(hq_{h+} - c(q_{h+}) + \omega hQ_{h+h} - C(Q_{h+h})] + (1 - \pi)((hq_{h-} - c(q_{h-}) + \omega lQ_{h-l} - C(Q_{h-l})] - U_h) \\
+ (1 - \nu)\{\pi[(lq_{l+} - c(q_{l+}) + \omega lQ_{l+l} - C(Q_{l+l})] + (1 - \pi)((lq_{l-} - c(q_{l-}) + \omega hQ_{l-h} - C(Q_{l-h})] - U_l). \}
\]

Hence, in the remainder we study the following maximization problem:

\[ \mathcal{P} : \max \Pi \text{ s.t. (8), (9), (10), (11)}. \]

We next characterize the solutions of this problem for all possible parameter constellations.

## 5 Profit-maximizing mechanisms

We start with considering the implementability of the perfect price discrimination outcome (1). Clearly, if a feasible direct mechanism is able to implement this outcome, then it must be profit-maximizing, as it maximizes the overall surplus and fully allocates this surplus to the seller. It would imply that the buyer’s private information does not limit the seller’s ability to extract rents so that the seller is able to attain first degree price discrimination.

While perfect price discrimination outcomes are usually not implementable in models of monopolistic screening, the following proposition shows that, in our model where the seller learns the persistence, perfect price discrimination is attainable when the degree of persistence, \( \pi \), is in between two thresholds \( \bar{\pi}_l^* \) and \( \bar{\pi}_h^* \), where\footnote{Note \( \bar{\pi}_l^* < 1/2 \), since \( \bar{\pi}_l^* = 1 - \frac{q_{l+} + \omega Q_{l+h}}{\omega Q_{l+} + Q_{l+h}} < 1 - \frac{\omega Q_{l+h}}{\omega Q_{l+} + Q_{l+h}} < 1 - \frac{\omega Q_{l+h}}{\omega Q_{l+} + Q_{l+h}} = 1/2. \)}

\[
\bar{\pi}_l^* \equiv 1 - \frac{q_{l+} + \omega Q_{l+h}}{\omega Q_{l+} + Q_{l+h}} < \bar{\pi}_h^* \equiv 1 - \frac{q_{h+} + \omega Q_{h+h}}{\omega Q_{l+} + Q_{l+h}} < 1.
\]

**Proposition 2** The seller attains perfect price discrimination if and only if \( \pi \in [\bar{\pi}_l^*, \bar{\pi}_h^*] \).
Indeed, as shown in Section 3, the outcome is not implementable with $\omega = 0$. In order to understand this difference, recall that in a classical framework, outcome (1) violates the incentive constraint of the efficient type $h$. The reason for this is that by claiming to be the inefficient type $\theta = l$, ex ante type $\theta = h$ can secure himself a strictly positive rent. Hence, the seller has to concede a strictly positive rent to induce type $\theta = h$ to reveal his type truthfully.

In a setup with negative persistence, a positive rent from understating one’s ex ante type is however not guaranteed, because when an ex ante type $\theta = h$ turns into an ex post inefficient type $\Theta = l$, the buyer is forced to overconsume, as he now receives the first best level of the ex post efficient type $\Theta = h$, which exceeds his first best level as an inefficient type $\Theta = l$. Hence, in a model with negative persistence, there is a force which dissuades an ex ante type $\theta = h$ from understating his type.

This force dominates when the degree of persistence is low enough in the sense that a negative persistence is sufficiently likely. This explains why a necessary condition for the implementability of the perfect price discrimination outcome is that the degree of persistence $\pi$ lies below the threshold level $\bar{\pi}_h$. On the other hand, when the degree of persistence $\pi$ is too low, the force is so strong that the seller now needs to concede an information rents to the ex ante inefficient type $\theta = l$. As a result, an implementability of outcome (1) also requires that the degree of persistence $\pi$ is not too low; it has to exceed the threshold value $\bar{\pi}_l$.

We next turn to analyzing the profit-maximizing mechanism when the negative persistence is so unlikely that, just as in classical models, the perfect price discrimination outcome violates the incentive constraint $(IC_h)$. Indeed, as reflected in our benchmark, it is well known that the binding constraints in monopolistic screening are the incentive constraint of the efficient type, $IC_h$, and the participation constraint of the inefficient type, $IR_l$.

The next proposition confirms that this property of monopolistic screening extends to our setup when the degree of persistence, $\pi$, not only exceeds the threshold $\bar{\pi}_h^*$ but also the threshold

$$\bar{\pi}_h^* = 1 - \frac{q_l^h + \omega Q_l^h}{\omega (Q_l^h + Q_l^h)},$$

where, similarly to $q_l^h$, the quantities $Q_l^h$ and $Q_l^h$ are implicitly defined by

$$C'(Q_l^h) = (l - \varphi \Delta)^+; C'(Q_l^h) = h + \varphi \Delta.$$

The next proposition fully characterizes the profit-maximizing outcome for this case.

**Proposition 3** If $\pi \geq \bar{\pi}_h^*$, then $\pi \geq \bar{\pi}_h^*$ and the solution exhibits both upward and downward distortions. At the optimum, only the incentive constraint $(IC_h)$ and the participation constraint $(IR_l)$ bind, and quantities satisfy

$$q_{l+} = q_{l-} = q_l^h < q_{h+} = q_{h-} = q_h^l; Q_{l+} = Q_{l-} = Q_l^h < Q_{h-} = Q_h^l < Q_{h+} = Q_h^* < Q_{l-} = Q_h^l.$$

The proposition shows that for the case $\pi \geq \bar{\pi}_h^*$, the optimal quantities $q_l$ and $q_h$ coincide with the

---

8The proposition may also seem to contradict findings in the literature of dynamic screening, which suggest that a seller must only concede information rents to the buyer for eliciting his ex ante private information (e.g., Esö and Szentes, 2017).
ones under monopolistic screening as derived in the benchmark of Proposition 1. That is, the output level for the high type $\theta = h$ is efficient, $q_h = q_h^*$, and, hence, there is no distortion at the top. In contrast, the output level for the low type $\theta = l$ is distorted downwards.

Moreover, the no-distortion at the top result extends to the quantity $Q$ as well. The ex ante high type $\theta = h$ receives also efficient quantities with respect to $Q$: the quantity $Q_h^*$ when he remains the high type and the quantity $Q_l^*$ when his type switches from high to low.

In contrast, the direction of the distortion for the ex ante low type $\theta = l$ depends on whether this type remains low, in which case the quantity is distorted downwards, $Q_{l+l} < Q_l^*$. By contrast, when the low type changes into a high type, the quantity is distorted upwards, $Q_{l-h} > Q_h^*$.

Since monopolistic screening models usually do not exhibit upward distortions, it is worthwhile to understand better the driving forces leading to the latter upward distortion. For this, recall that the optimal mechanism distorts quantities in order to reduce the information rents to the ex ante high type $\theta = h$ for truth-telling. To see that in our setup an upward rather than a downward distortion of $Q_{l-h}$ reduces these rents, suppose that $Q_{l-h} = Q_h^*$. This means that if the high type, $\theta = h$, claims to be low by sending a report $\tilde{\theta} = l$, he receives the quantity $Q_h^*$ in case of a negative persistence, where he actually changes into a low type, $\Theta = l$. However, from the perspective of a low type $\Theta = l$, the quantity $Q_{l-h} = Q_h^*$ is inefficiently high. Consequently, reducing $Q_{l-h}$ below $Q_h^*$ would only make the ex ante lie $\tilde{\theta} = l$ more attractive to type $\theta = h$. In contrast by increasing $Q_{l-h}$ beyond $Q_h^*$, the lie becomes less attractive and therefore reduces the information rents which the seller has to concede to the ex ante high type for revealing himself truthfully.

Propositions 3 fully characterize the outcome when $\tilde{\pi}^h \leq 0$, but leaves open the seller’s profit-maximizing contract for $\pi \in (\tilde{\pi}_h^*, \pi_h)$ in case $\pi_h > 0$. For these relative high degrees of persistence, the perfect price discrimination outcome violates the incentive constraint ($IC_h$), whereas the second best as characterized in proposition 3 violates the participation constraint ($IR_h$). This suggest that for these intermediate degrees of persistence, there are three binding constraints: ($IR_l$), ($IR_h$) and ($IC_h$). We confirm this suggestion next.

Proposition 4 If $\pi \in (\tilde{\pi}_h^*, \pi_h)$, then the solution exhibits both upward and downward distortions. At the optimum, the incentive constraint ($IC_h$) and both participation constraints ($IR_l$) and ($IR_h$) are binding, and quantities satisfy

$$q_{l+} = q_{l-} < q_l^* < q_{h-} = q_h^*; Q_{l+l} < Q_{h-l} = Q_l^* < Q_{h+h} = Q_h^* < Q_{l-h}.$$

In case that $\tilde{\pi}_l^* < 0$, the previous three propositions cover all possible degrees of persistence $\pi$. Since it holds $\tilde{\pi}_l^* < 0$ whenever $\omega < q_h^*/Q_l^*$, these are the cases for which the future is not too important.

We next extend our analysis to settings for which $\tilde{\pi}_l^* > 0$ so that $\pi$ may not exceed $\tilde{\pi}_l^*$. Recall that for $\pi < \tilde{\pi}_l^*$ outcome 1 violates the incentive constraint ($IC_l$) rather than ($IC_h$). Mirroring the idea behind Proposition 3 to investigate the optimal contract under the condition that the incentive constraint of the type that violates this outcome is binding together with the participation constraint of the other type, we first investigate contracts for which only the incentive constraint of the inefficient type, ($IC_l$), and the participation constraint of the efficient type, ($IR_h$) are binding.
Hence, the profit-maximizing mechanism is also robust to the extorsion possibility raised in footnote \footnote{6}. At the optimum, only the incentive constraint \((IC_i)\) and the participation constraint \((IR_h)\) bind and quantities satisfy

\[ q_{l+} = q_{l-} < q_{h+} < q_{h-} = q_{l-}'; Q_{h-l} = Q_i^l < Q_{l+l} = Q_i^h < Q_{l-h} = Q_h^l < Q_{h+h} = Q_h^l. \]

For \( \pi \in (\bar{\pi}^l, \bar{\pi}^l), \) the mechanisms identified in Proposition \footnote{2} are infeasible since they violate \((IC_i)\), whereas the mechanisms identified in Proposition \footnote{5} are infeasible since they violate \((IR_l)\). Mirroring the case analyzed in Proposition \footnote{4}, this suggests that for the range \( \pi \in (\bar{\pi}^l, \bar{\pi}^l), \) optimal contracts have all three constraints \((IR_l), (IC_i), (IR_h)\) binding. The following proposition confirms this suggestion and characterizes the profit-maximizing contract.

**Proposition 6** If \( \pi \in (\bar{\pi}^l, \bar{\pi}^l), \) then the solution exhibits both upward and downward distortions. At the optimum, the incentive constraint \((IC_i)\) and both participation constraints \((IR_l)\) and \((IR_h)\) are binding, and quantities satisfy

\[ q_{l+} = q_{l-} = q_i^l < q_{h+} = q_{h-} = q_i^h; Q_{h-l} = Q_i^l < Q_{l+l} = Q_i^h < Q_{l-h} = Q_h^l < Q_{h+h} = Q_h^l. \]

The previous propositions fully characterize the seller’s profit maximizing mechanisms for all possible parameter constellations. In particular, they deliver the following two insights.

First, it is not optimal to condition the first period quantity on the persistence \( p \). That is, it is optimal for the seller neither to differentiate between \( q_{l-} \) and \( q_{l+} \), nor to differentiate between \( q_{h-} \) and \( q_{h+} \). This property of the profit-maximizing mechanism first of all confirms the claim in Section \footnote{4} that the mechanism does not violate the presumed confidentiality of the seller’s report, since also the consumer cannot deduce it from observing \( q \). It, moreover, has the empirical implication that even though the quantities \( q \) depend on the degree of persistence \( \pi \), their levels are uninformative about whether the seller observed \( p = - \) or \( p = + \). Hence, from observing \( q \) an external observer can detect the buyer’s ex ante type but not whether there is persistence. Finally, this property of the profit-maximizing mechanism also motivates our claim in the modelling section that it does not matter whether the seller learns \( p \) before or after the buyer consumes the quantity \( q \). Because the optimal schedule does not condition the quantity \( q \), the seller is unaffected when it is physically impossible for her to condition the quantity \( q \) on learning about the buyer’s persistence, as is the case if stage 3 and 4 in our game were switched.

Second, whenever there are distortions, the quantities \( Q \) display both upward and downward distortions, whereas the quantities \( q \) display either an upward or downward distortion. More specifically,
the distortions concerning \( q \) depend on the degree of persistence \( \pi \); it exhibits downward distortions for high degrees of persistence and upward distortions for low degrees of persistence. Moreover, for high degrees of persistence, \( \pi > \pi^*_l \), the quantity \( Q_h \) of the ex ante high type \( \theta = h \) is, regardless of \( p \), efficient: \( Q_{h-} = Q^*_i \) and \( Q_{h+h} = Q^*_h \). In contrast, for low degrees of persistence, \( \pi < \pi^*_h \), the quantity \( Q_l \) of the low type \( \theta = l \) is, regardless of \( p \), efficient: \( Q_{l-} = Q^*_i \) and \( Q_{l+l} = Q^*_l \).

Figure 1 illustrates the optimal quantities as a function of the degree of persistence \( \pi \) and links the distortions of these optimal quantities with the specific ranges for which different combinations of constraints are binding. In particular, it illustrates the case when cost functions are quadratic and for which, as illustrated, the optimal quantities are piece-wise linear functions of \( \pi \).

6 Bigger Data

In this section we investigate the economic effects of a seller who becomes more and more sophisticated in using big data to learn about correlations. In order to capture this case, we use the parameter \( \omega \), which measures the relative importance of good \( Q \) versus good \( q \). Indeed, a small \( \omega \) represents a setting in which the seller’s data mining techniques have little economic impact, since mainly good \( q \) matters for the payoffs. However, as \( \omega \) grows large, good \( Q \) becomes more important, and the seller’s learning about the correlated structure becomes a driver of payoffs. This suggests that, as \( \omega \) grows, also the seller’s ability to price discriminate grows. Indeed, one may expect that, in the limit when \( \omega \) grows without bounds, the seller achieves an outcome arbitrarily close to the perfect price discrimination one. The main insight of this section is that, due to informational complementarities, this intuition is wrong. By contrast, the seller attains perfect price discrimination in the limit only when the degree of persistence \( \pi \) lies, once more, in an intermediate range, albeit a different one.

In order to obtain this insight, we introduce notation that relates the thresholds \( \tilde{\pi}^*_l \) and \( \tilde{\pi}^*_h \) to \( \omega \). In
particular, define the two thresholds $\omega^*_l$ and $\omega^*_h$ that correspond to the two thresholds $\bar{\pi}^*_l$ and $\bar{\pi}^*_h$:

$$\omega^*_l \equiv \frac{q^*_l}{Q^*_l - \pi(Q^*_l + Q^*_h)}; \quad \omega^*_h \equiv \frac{q^*_h}{Q^*_h - \pi(Q^*_h + Q^*_l)}.$$ 

It then follows that $\omega^*_l$ is strictly positive if and only if

$$\pi < \bar{\pi}^*_l \equiv \frac{Q^*_l}{Q^*_l + Q^*_h} = \lim_{\omega \to \infty} \bar{\pi}^*_l.$$ 

Likewise, $\omega^*_h$ is strictly positive if and only if

$$\pi < \bar{\pi}^*_h \equiv \frac{Q^*_h}{Q^*_h + Q^*_l} = \lim_{\omega \to \infty} \bar{\pi}^*_h.$$ 

It is straightforward to see that the limit values $\bar{\pi}^*_l$ and $\bar{\pi}^*_h$ satisfy the following ordering:

$$0 < \bar{\pi}^*_l < 1/2 < \bar{\pi}^*_h < 1.$$ 

Equipped with this notation, the next proposition shows that only if the degree of persistence, $\pi$, lies in between these two limit values, then perfect price discrimination is attainable when $\omega$ grows unbounded.

**Proposition 7** For the limit case, where $\omega$ grows unbounded, the profit-maximizing outcome coincides with perfect price discrimination if and only if $\pi \in [\bar{\pi}^*_l, \bar{\pi}^*_h]$.

The proposition shows that, even if $\omega$ becomes unboundedly large so that the economic significance of the good $q$ vanishes, learning the buyer’s persistence does not enable the seller to extract all rents from the buyer, except for intermediate degrees of persistence. We obtain this result even though, as argued in Section 4, the seller can ensure a truthful revelation of both $p$ and $\Theta$ at no additional costs in excess of the costs associated with a truthful revelation of the buyer’s ex ante private information $\theta$. This means that the seller has to concede information rents to the buyer only for his ex ante private information concerning his valuation of the good $q$. The proposition shows that these information rents do not vanish when the good $q$ becomes economically insignificant.

The fact that the buyer’s ex ante private information $\theta$ remains valuable even if the consumption value of the good $q$ becomes insignificant depends on an informational complementarity. Indeed, from only learning whether the agent’s persistence is positive or negative the seller cannot fully deduce the value of $\Theta$, except for the trivial cases $\pi = 0$ or $\pi = 1$. It is only in combination with the buyer’s ex ante private information that learning the persistence allows the seller to learn $\Theta$ perfectly.

Hence, in our setting the buyer’s ex ante private information has two informational roles. Its first role is the usual one of restricting the seller in extracting rents concerning the consumption of good $q$. Its secondary role is completing the seller’s information about $p$ and thereby allowing to learn $\Theta$ perfectly.

In line with standard intuition, the first informational role of the buyer’s ex ante private information

10That is, $\pi = \bar{\pi}^*_l \iff \omega = \omega^*_l$. 

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This is most easily seen after rewriting \( \bar{\pi} \) where the complementarity is extreme. Note that without observing \( \pi \), the seller observes both \( p \) remain unchanged, indicating that this perfect learning is completely uninformative. In contrast, if \( \Theta \) are equally likely and the seller's uncertainty is largest. In contrast, for the extremes \( \Theta = \bar{\pi} \), the probability of \( \Theta = h \) equals \( \pi(1 - \nu) + (1 - \pi)\nu \). Given \( \nu = 1/2 \), both probabilities are 1/2. Now compare this case to the one in which the seller observes \( p \). If she observes \( p = + \), she knows that \( \Theta = h \) if and only if \( \theta = h \). Since the likelihood of \( \theta = h \) is \( \nu = 1/2 \), she puts probability 1/2 on \( \Theta = h \) after observing \( p = + \). Similarly, if she observes \( p = - \), she now knows that \( \Theta = h \) if and only if \( \theta = l \). But since the likelihood of \( \theta = l \) is \( 1 - \nu = 1/2 \), she also puts probability 1/2 on \( \Theta = h \) after observing \( p = - \). Hence, despite learning the realization of \( p \) perfectly, her beliefs about \( \Theta \) remain unchanged, indicating that this perfect learning is completely uninformative. In contrast, if the seller observes both \( p \) and \( \theta \), she is certain that \( \Theta = h \) if \( p = + \) and \( \theta = l \), or if \( \Theta = l \) if \( p = - \), while she is certain that \( \Theta = l \) for the other realizations of \( p \) and \( \theta \). Hence, perfectly learning \( p \) is informative about \( \Theta \), and perfectly so, only if the seller also learns \( \theta \), whereas it is perfectly uninformative if the seller does not learn \( \theta \). This illustrates the informational complementary between \( p \) and \( \theta \).

Indeed, the informational complementarity is strongest in the case \( \nu = 1/2 \), where both valuations are equally likely and the seller's uncertainty is largest. In contrast, for the extremes \( \nu = 0 \) and \( \nu = 1 \), the seller knows the buyer's ex ante type so that the buyer has no ex ante private information. As a result, a seller learning the buyers' correlation enables her to learn the buyer's ex post valuation perfectly so that she can implement the first best while extracting all rents. However, since she could also attain this first best solution without learning the correlation, also in this case learning the correlation has no real economic value to her.

The formal proof of the Proposition 7 is straightforward and depends on the fact that the thresholds \( \bar{\pi}_l^* \) and \( \bar{\pi}_h^* \) are increasing in \( \omega \). In order to obtain further insights about the implementability of perfect price discrimination, it is instructive to consider the comparative statics in \( \omega \) for the three separate cases: 1) \( \pi \) is intermediate, 2) \( \pi \) exceeds the threshold \( \bar{\pi}_h^\infty \); 3) \( \pi \) is smaller than the threshold \( \bar{\pi}_l^\infty \). Figure 2 illustrates these cases graphically.

Starting with the immediate case, note that the threshold \( \omega_l^* \) is negative so that \( \pi \) exceeds \( \bar{\pi}_l^* \) for all \( \omega \). In contrast, \( \omega_h^* \) is positive so that for \( \omega \) smaller than the threshold \( \omega_h^* \), we have that \( \pi \) exceeds

\[ \bar{\pi}_l^\infty = \frac{Q_l^*}{Q_l^* + Q_h^*}, \quad \bar{\pi}_h^\infty = \frac{Q_h^*}{Q_l^* + Q_h^*}. \]
\(\bar{\pi}_h^*,\) implying that the seller cannot attain perfect price discrimination. In contrast, when \(\omega\) exceeds the threshold \(\omega_h^*\), we have \(\pi\) smaller than \(\bar{\pi}_h^*\) but larger than \(\bar{\pi}_i^*\). Proposition 2 then implies that perfect price discrimination is implementable. As a consequence, we obtain perfect price discrimination not only at the limit but for any finite value of \(\omega\) exceeding \(\omega^*_h\).

For the case, where \(\pi\) exceeds the threshold \(\bar{\pi}_h^\infty\), we have that both \(\omega_l^*\) and \(\omega_h^*\) are negative so that \(\pi\) exceeds both \(\bar{\pi}_l^*\) and \(\bar{\pi}_h^*\) for any \(\omega\). Hence, for this case either Proposition 3 or 4 apply, implying that the outcome of perfect price discrimination is unattainable for any \(\omega\), including the limit.

Finally, in the case that the threshold \(\bar{\pi}_l^\infty\) exceeds \(\pi\), we have that both \(\omega_l^*\) and \(\omega_h^*\) are positive and, moreover, \(\omega_l^*\) exceeds \(\omega_h^*\). Hence, for \(\omega\) smaller than \(\omega_h^*\), we have \(\pi\) larger than \(\bar{\pi}_h^*\), implying that the outcome of perfect price discrimination is unattainable, as either Proposition 3 or 4 apply. For \(\omega\) lying in between \(\omega_l^*\) and \(\omega_h^*\), we have that \(\pi\) lies in between \(\bar{\pi}_l^*\) and \(\bar{\pi}_h^*\), implying that Proposition 2 applies so that the outcome of perfect price discrimination is attainable. Yet, for \(\omega\) exceeding \(\omega_l^*\), both \(\bar{\pi}_h^*\) and \(\bar{\pi}_i^*\) exceed \(\pi\), implying that the outcome of perfect price discrimination is unattainable for \(\omega > \omega_l^*\) since either Proposition 5 or 6 apply. In this last case, the comparative statics are non-monotonic. Interestingly, perfect price discrimination is attainable for intermediate values of \(\omega\) but not in the limit where \(\omega\) grows unbounded.

Because the parameter \(\nu\) measures the amount of uncertainty of the seller about the buyer's ex ante type \(\theta\), it affects the strength of the complementarity effect. For this reason, it is insightful to study also the model's comparative statics with respect to the parameter \(\nu\). We do so in the next lemma.

**Lemma 2** The thresholds \(\bar{\pi}_l^*\) and \(\bar{\pi}_h^*\) are independent of \(\nu\). The threshold \(\bar{\pi}_h^*\) is increasing in \(\nu\) whenever \(\bar{\pi}_h^* > 0\), it equals \(\bar{\pi}_h^*\) for \(\nu = 0\), and equals 1 for \(\nu \geq \bar{\nu}_h\), where \(\bar{\nu}_h < 1\). The threshold \(\bar{\pi}_l^*\) is increasing in \(\nu\) whenever \(\bar{\pi}_l^* > 0\), it equals \(\bar{\pi}_l^*\) for \(\nu = 1\), and is smaller than 0 for \(\nu < \bar{\nu}_l\), where \(\bar{\nu}_l \in (0, 1)\).

Figure 3 illustrates the result for the case that \(\bar{\pi}_l^*, \bar{\pi}_h^* > 0\). In addition, it shows how the different cases covered in the Propositions 2-6 associate with the different combinations of \(\nu\) and \(\pi\).
7 Conclusion

Monopolistic screening models show that a buyer’s ex ante private information represents a countervailing power against the monopolist’s ability to extract all gains from trade, guaranteeing buyers a positive information rent. For welfare perspectives that focus on consumer’s well-being such as consumer surplus, this reflects an important insight. Our analysis shows that, due to an informational complementarity, these rents persist if the buyer faces a correlation-savvy monopolist who perfectly learns about the correlation of his valuations, and the degree of persistence is large enough. This inability of the monopolist to achieve perfect price discrimination persists even if the economic significance of the consumption value of the good about which the consumer possesses his ex ante private information vanishes. Similarly to monopolistic screening, the inability of the correlation-savvy monopolist to extract all gains from trade, induces her to distort quantities from their efficient values. However, contrary to standard monopolistic screening models, which display only downward distortions, a correlation-savvy seller optimally distorts quantities both downward and upward.

A more general conclusion of our analysis is to caution about viewing learning about correlations and learning about a buyer’s private information as equivalent and naively extrapolating from the one context to the other. As argued, economic results differ qualitatively between a model in which a monopolist learns about the consumer’s private information directly and where she learns only about correlations. Hence, in the context of big data, it matters whether retailers use data analysis for identifying robust correlations, or for learning about the private information of specific customers. Importantly, this difference then also affects the optimal regulatory response to these data mining activities.

Given the stylized nature of our model, it is useful to discuss the role of its underlying assumptions in arriving at our specific results and for also keeping the analysis tractable.

First, we assumed that the monopolist can learn the buyer’s correlation perfectly. This assumption does not only yield a more tractable model than one in which the monopolist learns the correlation only imperfectly; it also represents the monopolist’s best possible case. Since we have shown that even with perfect learning, the monopolist is generally unable to attain perfect price discrimination, this result is robust to assuming that a monopolist can learn correlations only imperfectly. Moreover, the extreme assumption of perfect learning also underscores the informational complementarity in a setup with learning about correlations: despite learning the correlation perfectly, the information may be useless if the seller does not induce the buyer to reveal his ex ante type.

We further considered a setup in which the monopolist observes the correlation privately. It however directly follows from our analysis that identical results obtain when this correlation is revealed publicly (for instance by some third party such as a platform). The reason is that, as explained in Section 4, the seller can exploit the correlation between her private information about persistence and the buyer’s ex post private information to induce truthtelling about these variables without any further costs. As a result, the model is as if the observed persistence is verifiable and the monopolist can directly condition her contract in this information. For this reason, a model in which the buyer’s persistence is learned publicly yields identical results. This also means that the monopolist does neither gain nor lose from learning the persistence in private; she has no incentive to hide this information.

12In contrast, ex post private information does not prevent a monopolist from extracting all rents and does therefore not represent a countervailing power to monopoly power.
We assumed that the seller learns about the correlation structure only after she offers the contract, but before the buyer consumes the first good \( q \). If the seller’s learning takes place before she offers a contract, a signalling game ensues, leading to the well-known problem of multiple equilibrium outcomes. However, our previous comment that identical results obtain when assuming that this correlation is revealed publicly implies both that our equilibrium outcome remains an equilibrium outcome in the signalling game, and also that it yields the seller the highest profit. As a result, the seller is also unable to attain perfect price discrimination in some signalling game version of our setup. Moreover, applying the refinement as proposed in In and Wright (2018) which is based on this exact reordering of moves, the equilibrium outcome that we obtained in our analysis is the unique equilibrium outcome in the signalling game that satisfies this refinement. In addition, given that the profit-maximizing mechanism does not exploit the possibility that the initial quantity \( q \) depends on the correlation structure shows that this outcome is also attainable if the seller observes the correlation structure only after the buyer consumes \( q \).

Assuming only binary valuations helped to keep the model tractable. With binary valuations, the correlation of types boils down to the question whether the type is persistent. I.e., does the valuation switch or remain the same? A more elaborate model with more than two types is richer in the sense that learning about persistence, i.e. learning whether a type does or does not switch, is not a sufficient statistic, because in the case of switching it is unclear to which of the available other values the valuation switches. In other words, with more than binary types, the dimensionality of the transition matrix increases. In particular, an \( n \times m \) transition matrix, has \( m^n \) possible deterministic transitions so that complexity rises dramatically. Yet, our binary analysis reveals that the driving force behind whether the monopolist can attain first degree price discrimination is a trade-off between negative and positive correlation. I.e., is the type likely to improve or to worsen? This qualitative feature is best captured in a binary setup but, of course, also holds in a setup with more than two types. In this case, it then however also matters how much a type worsens or improves, and we expect that such magnitudes will also play a role in determining the exact conditions under which first degree price discrimination is attainable.

Lastly, we analyzed the monopolist’s profit-maximizing outcome assuming that she can commit to one overall contract specifying the terms of trade of both goods \( q \) and \( Q \). Also this simplifies the analysis, because it still enables a revelation principle despite of the limited commitment of the seller concerning reporting her observed persistence. Without the ability to commit to one overall contract, direct mechanisms are no longer without loss and the model’s tractability is lost. However, similar to the assumption about perfect learning, the monopolist’s full commitment represents a best case scenario. That is, our main result that a correlation-savvy seller is, in general, unable to perfectly price discriminate is also robust with respect to the monopolist’s contractual commitment. At best the monopolist can do just as well with less commitment but not strictly better. However, without the ability to commit to one overall contract, her optimal way to distort allocations may differ, because these distortions can be expected to be also partially driven by the monopolist’s limited contractual commitment. Since, in the context of a multi-product monopolist, limited contractual commitment seems natural but within our specific modelling setup analytically not tractable, we leave it for future research.
Appendix

This appendix collects the proofs of the propositions and lemmas in the main text.

Proof of Proposition[1] By the revelation principle, the profit-maximizing mechanism is an incentive compatible mechanism \( \{(q_l, p_l), (q_h, p_h)\} \), which is individual rational and maximizes the monopolist’s expected profits \( \Pi \). That is, it is a combination \((q_l, p_l, q_h, p_h)\) that maximizes \( \Pi \) subject to (2) and (3).

Considering the relaxed problem with only \((IC_h)\) and \((IR_l)\) yields that, for this relaxed problem, both constraints are binding, implying \( p_l = lq_l \) and \( p_h = hq_h - q_l \Delta \). After a substitution of these prices in the objective function, we obtain as maximizers \( q_h = q_h^l \) and \( q_l = q_l^l \). It is straightforward to check that the implied direct mechanism satisfies the neglected constraints \((IC_l)\) and \((IR_h)\). Q.E.D.

Proof of Lemma[1] We argue that for any direct mechanisms \( \gamma = (q_{\hat{\theta}h}, Q_{\hat{\theta}h}, T_{\hat{\theta}h}) \) that is feasible, we can find a combination \((Q_{h-h}, Q_{l-l}, Q_{h+l}, Q_{l+l}, T_{h-h}, T_{l-l}, T_{h+h}, T_{l+l})\) so that the adapted direct mechanism is payoff equivalent to the original one, is feasible, and it is Bayesian incentive compatible for the seller to report \( p \) truthfully honestly and the buyer to report \( \Theta \) truthfully. Payoff-equivalence of the adaptation is trivially ensured since given truthful reporting, the combination \((Q_{h-h}, Q_{l-l}, Q_{h+l}, Q_{l+l}, T_{h-h}, T_{l-l}, T_{h+h}, T_{l+l})\) does not affect the buyer’s or the seller’s payoff. Because of this, feasibility also trivially follows since the combination does also not affect constraints \((IC_h)\), \((IC_l)\), \((IR_h)\) and \((IR_l)\).

It remains to show that, as claimed in footnote [6] we can find a combination \((Q_{h-h}, Q_{l-l}, Q_{h+l}, Q_{l+l}, T_{h-h}, T_{l-l}, T_{h+h}, T_{l+l})\) such that it is optimal for the seller to report \( p \) truthfully if she believes that the buyer reports honestly. In the case of positive persistence, this requires

\[
\nu \Pi(T_{h+h}, q_{h+h}, Q_{h+h}) + (1 - \nu) \Pi(T_{l+l}, q_{l+l}, Q_{l+l}) \geq \nu \Pi(T_{h-h}, q_{h-h}, Q_{h-h}) + (1 - \nu) \Pi(T_{l-l}, q_{l-l}, Q_{l-l});
\]

and in the case of negative persistence, this requires

\[
\nu \Pi(T_{h-l}, q_{h-l}, Q_{h-l}) + (1 - \nu) \Pi(T_{l+h}, q_{l+h}, Q_{l+h}) \geq \nu \Pi(T_{h+h}, q_{h+h}, Q_{h+h}) + (1 - \nu) \Pi(T_{l+l}, q_{l+l}, Q_{l+l}).
\]

At the same time, it must be optimal for the buyer of ex ante type \( \theta \) to report \( \Theta \) truthfully if he reported \( \theta \) honestly and believes that the seller reports the persistence honestly. For ex ante type \( \theta = h \), this is the case if the following three inequalities are met

\[
\pi U(T_{h+h}, q_{h+h}, Q_{h+h}|h, h) + (1 - \pi) U(T_{h-l}, q_{h-l}, Q_{h-l}|h, l) \geq
\pi U(T_{h+l}, q_{h+l}, Q_{h+l}|h, h) + (1 - \pi) U(T_{h-l}, q_{h-l}, Q_{h-l}|h, l);
\]

\[
\pi U(T_{h+h}, q_{h+h}, Q_{h+h}|h, h) + (1 - \pi) U(T_{h-l}, q_{h-l}, Q_{h-l}|h, l) \geq
\pi U(T_{h+h}, q_{h+h}, Q_{h+h}|h, h) + (1 - \pi) U(T_{h-h}, q_{h-h}, Q_{h-h}|h, l);
\]

\[
\pi U(T_{h+h}, q_{h+h}, Q_{h+h}|h, h) + (1 - \pi) U(T_{h-l}, q_{h-l}, Q_{h-l}|h, l) \geq
\pi U(T_{h+l}, q_{h+l}, Q_{h+l}|h, h) + (1 - \pi) U(T_{h-l}, q_{h-l}, Q_{h-l}|h, l).
\]
For ex ante type \( \theta = l \), this is the case if the following three inequalities are met

\[
\pi U(T_{i+l}, q_{l+i}, Q_{l+i}|l, l) + (1 - \pi) U(T_{l-h}, q_{l-h}, Q_{l-h}|l, h) 
\geq \pi U(T_{i+l}, q_{l+i}, Q_{l+i}|l, l) + (1 - \pi) U(T_{l-h}, q_{l-h}, Q_{l-h}|l, h); \\
\pi U(T_{i+l}, q_{l+i}, Q_{l+i}|l, l) + (1 - \pi) U(T_{l-h}, q_{l-h}, Q_{l-h}|l, h) 
\geq \pi U(T_{i+l}, q_{l+i}, Q_{l+i}|l, l) + (1 - \pi) U(T_{l-h}, q_{l-h}, Q_{l-h}|l, h); \\
\pi U(T_{i+l}, q_{l+i}, Q_{l+i}|l, l) + (1 - \pi) U(T_{l-h}, q_{l-h}, Q_{l-h}|l, h) 
\geq \pi U(T_{i+l}, q_{l+i}, Q_{l+i}|l, l) + (1 - \pi) U(T_{l-h}, q_{l-h}, Q_{l-h}|l, h). 
\]

Each of these 8 inequalities is implied if the following 8 conditions on the state-by-state utilities hold simultaneously

\[
\Pi(T_{h+h}, q_{h+h}, Q_{h+h}) \geq \Pi(T_{h-h}, q_{h-h}, Q_{h-h}) \text{ and } \Pi(T_{i+i}, q_{i+i}, Q_{i+i}) \geq \Pi(T_{i-i}, q_{i-i}, Q_{i-i}); \\
\Pi(T_{h-i}, q_{h-i}, Q_{h-i}) \geq \Pi(T_{h+i}, q_{h+i}, Q_{h+i}) \text{ and } \Pi(T_{l-h}, q_{l-h}, Q_{l-h}) \geq \Pi(T_{l+h}, q_{l+h}, Q_{l+h}); \\
U(T_{h+h}, q_{h+h}, Q_{h+h}|h, h) \geq U(T_{h+i}, q_{h+i}, Q_{h+i}|h, h) \text{ and } U(T_{h-i}, q_{h-i}, Q_{h-i}|h, h) \geq U(T_{h-h}, q_{h-h}, Q_{h-h}|h, h); \\
U(T_{i+i}, q_{i+i}, Q_{i+i}|l, l) \geq U(T_{i+h}, q_{i+h}, Q_{i+h}|l, l) \text{ and } U(T_{i-i}, q_{i-i}, Q_{i-i}|l, h) \geq U(T_{i-l}, q_{i-l}, Q_{i-l}|l, h). 
\]

Hence, it suffices to show that for any 8 numbers \((K_1, \ldots, K_8)\), we find a combination \((Q_{h-h}, Q_{l-i}, Q_{h+h}, Q_{l+i}, T_{h-I}, T_{l+i}, T_{h+i}, T_{h-i})\) such that

\[
K_1 \geq \Pi(T_{h-h}, q_{h-h}, Q_{h-h}) \text{ and } K_2 \geq \Pi(T_{l-i}, q_{l-i}, Q_{l-i}) \\
K_3 \geq \Pi(T_{h+i}, q_{h+i}, Q_{h+i}) \text{ and } K_4 \geq \Pi(T_{l+h}, q_{l+h}, Q_{l+h}); \\
K_5 \geq U(T_{h+i}, q_{h+i}, Q_{h+i}|h, h) \text{ and } K_6 \geq U(T_{h-h}, q_{h-h}, Q_{h-h}|h, h); \\
K_7 \geq U(T_{i+h}, q_{i+h}, Q_{i+h}|l, l) \text{ and } K_8 \geq U(T_{i-l}, q_{i-l}, Q_{i-l}|l, h). 
\]

To show this is indeed the case, we first regroup these 8 inequalities, as follows:

\[
K_1 \geq \Pi(T_{h-h}, q_{h-h}, Q_{h-h}) \text{ and } K_6 \geq U(T_{h-h}, q_{h-h}, Q_{h-h}|h, l); \\
K_2 \geq \Pi(T_{l-i}, q_{l-i}, Q_{l-i}) \text{ and } K_8 \geq U(T_{l-i}, q_{l-i}, Q_{l-i}|l, h); \\
K_3 \geq \Pi(T_{h+i}, q_{h+i}, Q_{h+i}) \text{ and } K_5 \geq U(T_{h+h}, q_{h+h}, Q_{h+h}|h, h); \\
K_4 \geq \Pi(T_{l+i}, q_{l+i}, Q_{l+i}) \text{ and } K_7 \geq U(T_{l+i}, q_{l+i}, Q_{l+i}|l, l). 
\]

so that taking each pair of inequalities together implies the following four necessary conditions:

\[
K_1 + K_6 \geq \Pi(T_{h-h}, q_{h-h}, Q_{h-h}) + U(T_{h-h}, q_{h-h}, Q_{h-h}|h, l) = \theta q_{h-h} - c(q_{h-h}) + \omega(\Theta Q_{h-h} - C(Q_{h-h})); \\
K_2 + K_8 \geq \Pi(T_{l-i}, q_{l-i}, Q_{l-i}) + U(T_{l-i}, q_{l-i}, Q_{l-i}|l, h) = \theta q_{l-i} - c(q_{l-i}) + \omega(\Theta Q_{l-i} - C(Q_{l-i})); \\
K_3 + K_5 \geq \Pi(T_{h+i}, q_{h+i}, Q_{h+i}) + U(T_{h+i}, q_{h+i}, Q_{h+i}|h, h) = \theta q_{h+i} + c(q_{h+i}) + \omega(\Theta Q_{h+i} - C(Q_{h+i})); \\
K_4 + K_7 \geq \Pi(T_{l+i}, q_{l+i}, Q_{l+i}) + U(T_{l+i}, q_{l+i}, Q_{l+i}|l, l) = \theta q_{l+i} - c(q_{l+i}) + \omega(\Theta Q_{l+i} - C(Q_{l+i})). 
\]

The convexity of \( C(Q) \) and the assumption \( \lim_{Q \to \infty} C'(Q) = \infty \) imply that for any combi-
nation \((K_1, \ldots, K_8)\) and \((q_{-}, q_{t}, q_{h}, q_{l})\) and \(\omega > 0\), these four inequalities are satisfied for \((Q_{t-h}, Q_{t-l}, Q_{h+l}, Q_{l+h})\) large enough. We may then also find constants \((T_{t-h}, T_{t-l}, T_{h+l}, T_{l+h})\) so that together with these large \((Q_{t-h}, Q_{t-l}, Q_{h+l}, Q_{l+h})\) each of the previous 8 inequalities holds. This completes the proof and also shows the claim in footnote [6].

Q.E.D.

Proof of Proposition 2: We verify that the efficient quantities together with \(U_h = U_l = 0\) satisfy the incentive constraints \(IC_h\) and \(IC_l\) if and only if \(\pi \in [\bar{\pi}_h, \bar{\pi}_h]\).

First, note that \(IC_h\) associated with this outcome simplifies to

\[
\pi[q_h^* + \omega Q_t^*] \Delta + (1 - \pi)[q_t^* - \omega Q_h^*] \Delta \leq 0
\]

which is equivalent to \(\pi \leq \bar{\pi}_h\).

Next, note that \(IC_l\) associated with the first best simplifies to

\[
0 \geq -\pi[q_h^* + \omega Q_t^*] - (1 - \pi)[q_t^* - \omega Q_h^*]
\]

which is equivalent to \(\pi \geq \bar{\pi}_h\).

The proposition follows from noting that \(q_h^* > q_t^*\) and \(Q_h^* > Q_t^*\) implies \(\bar{\pi}_l > \bar{\pi}_h\). Q.E.D.

Proof of Proposition 3: We first prove that \(\pi \geq \bar{\pi}_h\) implies \(\pi \geq \bar{\pi}_h\). Note that, due to \(\pi \geq 0\), this trivially holds when \(\bar{\pi}_h < 0\) and \(\bar{\pi}_h < 0\). So suppose \(\bar{\pi}_h \geq 0\) or \(\bar{\pi}_h > 0\) holds. The first inequality implies that \(\omega \geq \omega^h = q_t^h / Q_t^h\), whereas the second implies that \(\omega \geq \omega_h^* = q_t^* / Q_t^*\). Note that \(\omega^h < \omega_h^*\) since \(q_t^h < q_t^*\) and \(Q_t^h > Q_t^*\). Because both \(\bar{\pi}_h\) and \(\bar{\pi}_h^*\) are increasing in \(\omega\), we have that \(\bar{\pi}_h \geq 0\) implies \(\bar{\pi}_h \geq 0\). Hence, the statement \(\bar{\pi}_h \geq 0\) or \(\bar{\pi}_h > 0\) implies \(\omega \geq \omega^h\).

We next show that for \(\omega \geq \omega^h\), we have \(\bar{\pi}_h > \bar{\pi}_h^*\). To see this, define

\[
\Delta^h(\omega) \equiv \bar{\pi}_h^* - \bar{\pi}_h^* = q_t^* + \omega Q_t^* - \left(q_t^* + \omega Q_h^*\right) = \frac{(q_t^* + \omega Q_t^* - q_t^* + \omega Q_h^*)}{(Q_h^* + Q_t^*)}
\]

so that it follows

\[
\frac{\partial \Delta^h}{\partial \omega} = \frac{q_t^* (Q_h^* + Q_t^*) - q_t^* (Q_h^* + Q_t^*)}{(Q_h^* + Q_t^*)^2}.
\]

Hence, the sign of \(\partial \Delta^h / \partial \omega\) does not depend on \(\omega\). If the sign of \(\partial \Delta^h / \partial \omega\) is positive, then recall that at \(\omega = \omega^h\) we have \(\bar{\pi}_h^* = 0\) and \(\bar{\pi}_h > 0\) so that \(\Delta^h(\omega) = 0 - \bar{\pi}_h > 0\). The positive sign of \(\partial \Delta^h / \partial \omega\) then implies \(\Delta^h(\omega) > 0\) for all \(\omega \geq \omega^h\) and, hence, \(\bar{\pi}_h > \bar{\pi}_h^*\).

If the sign of \(\partial \Delta^h / \partial \omega\) is negative, the result \(\bar{\pi}_h \geq \bar{\pi}_h^*\) then follows from

\[
\Delta^h(\omega) \geq \lim_{\omega \to -\infty} \Delta^h(\omega) = \frac{Q_t^* (Q_h^* + Q_t^*) - Q_t^* (Q_h^* + Q_t^*)}{(Q_h^* + Q_t^*)^2} = \frac{Q_t^* Q_h^* - Q_t^* Q_h^*}{(Q_h^* + Q_t^*)^2} > 0,
\]

where the last inequality follows from \(Q_t^* > Q_t^* > 0\) and \(Q_t^* > Q_t^* > 0\).

Hence, \(\pi \geq \bar{\pi}_h\) implies \(\pi \geq \bar{\pi}_h\) so that the first best violates \(IC_h\). We now show that \(\pi \geq \bar{\pi}_h\) implies that only \((IC_h)\) and \((IC_l)\) bind at the maximum. Indeed, a binding incentive constraint \((IC_h)\)
and participation constraint \((IR_l)\) imply

\[
U_l = 0; U_h = \pi[q_{l+} + \omega Q_{l+l}]\Delta + (1 - \pi)[q_{l-} - \omega Q_{l-h}]\Delta.
\]

Substitution of these values for \(U_h\) and \(U_l\) in \(\Pi\), implies maximizing

\[
\begin{align*}
&\nu\{\pi[(lq_{h+} - c(q_{h+})] + \omega(hQ_{h+h} - C(Q_{h+h}))\} + (1 - \pi)[(lq_{h-} - c(q_{h-})] + \omega(lQ_{h-l} - C(Q_{h-l}))\} \\
&+ (1 - \nu)\nu\{[(l - \varphi)Q_{l+} - c(q_{l+})] + \omega((l - \varphi)Q_{l+l} - C(Q_{l+l}))\} \\
&+ (1 - \nu)(1 - \pi)[((l - \varphi)Q_{l-} - c(q_{l-})] + \omega((h + \varphi)Q_{l-h} - C(Q_{l-h}))\} \\
\end{align*}
\]

with solution \(q_{l+} = q_{l-} = q_h^h < q_{h+} = q_{h-} = q_h^s, Q_{l+l} = q_l^h < Q_{h-l} = Q_l^l < Q_{h+h} = Q_h^s < Q_{l-h} = Q_l^h\). To see that for this solution, \((IC_l)\) holds, note that for this solution \((IC_l)\) simplifies to

\[
0 \geq \pi[q_l^h + \omega Q_l^h]\Delta + (1 - \pi)[q_l^h - \omega Q_l^h]\Delta - \pi[q_h^s + \omega Q_h^s]\Delta - (1 - \pi)[q_h^s - \omega Q_l^h]\Delta,
\]

and rewrites as

\[
\pi[q_h^s - q_l^h + \omega(Q_h^s - Q_l^h)] + (1 - \pi)[q_h^s - q_l^h + \omega(Q_h^h - Q_l^h)] \geq 0,
\]

which holds since \(q_h^s > q_l^h\) and \(Q_h^s > Q_l^s > Q_l^h\).

Moreover, the solution satisfies the neglected \((IR_h)\) if and only if

\[
U_h = \pi[q_l^h + \omega Q_l^h]\Delta + (1 - \pi)[q_l^h - \omega(Q_h^h + Q_l^h)]\Delta = (q_l^h + \omega Q_l^h)\Delta - (1 - \pi)\omega(Q_h^h + Q_l^h)\Delta \geq 0,
\]

which if equivalent to \(\pi \geq \bar{\pi}^h\).

\[\text{Q.E.D.}\]

**Proof of Proposition**

Assuming the three binding constraints are \((IR_l)\), \((IR_h)\) and \((IC_h)\), we have

\[
U_l = 0; U_h = 0; 0 = \pi[q_{l+} + \omega Q_{l+l}]\Delta + (1 - \pi)[q_{l-} - \omega Q_{l-h}]\Delta.
\]

Substitution of \(U_h = U_l = 0\) implies to maximize

\[
W_3 \equiv \nu\{\pi[(lq_{h+} - c(q_{h+})] + \omega(hQ_{h+h} - C(Q_{h+h}))\} + (1 - \pi)[(lq_{h-} - c(q_{h-})] + \omega(lQ_{h-l} - C(Q_{h-l}))\} \\
+ (1 - \nu)\nu\{[(lq_{l+} - c(q_{l+})] + \omega(lQ_{l+l} - C(Q_{l+l}))\} + (1 - \pi)[[(lq_{l-} - c(q_{l-})] + \omega(lQ_{h-l} - C(Q_{l-l}))\} \\
s.t. (1 - \nu)[\pi[q_{l+} + \omega Q_{l+l}] + (1 - \pi)[q_{l-} - \omega Q_{l-h}] = 0.
\]

The associated Lagrangian is

\[
\mathcal{L} \equiv \nu\{\pi[(lq_{h+} - c(q_{h+})] + \omega(hQ_{h+h} - C(Q_{h+h}))\} + (1 - \pi)[(lq_{h-} - c(q_{h-})] + \omega(lQ_{h-l} - C(Q_{h-l}))\} \\
+ (1 - \nu)\nu\{[(l - \lambda)q_{l+} - c(q_{l+})] + \omega((l - \lambda)Q_{l+l} - C(Q_{l+l}))\} \\
+ (1 - \pi)[((l - \lambda)q_{l-} - c(q_{l-})] + \omega((h + \lambda)Q_{l-h} - C(Q_{l-h}))\},
\]

where \(\lambda\) is the lagrange multiplier. Hence, the optimality conditions w.r.t. \(q_{h+}, Q_{h+h}, q_{h-}, Q_{h-l}\) imply \(q_{h+} = Q_{h+h} = q_{h-} = q_h^s, \text{ and } Q_{h-l} = q_l^l\); the optimality conditions w.r.t. \(q_{l+}, Q_{l+l}, q_{l-}\) coincide; they
satisfy
\[ c'(q_{l+}) = C'(Q_{l+h}) = c'(q_{l-}) = (l - \lambda)^+; \]
where for \( Q_{l-h} \) it optimally holds
\[ C'(Q_{l-h}) = (h + \lambda)^+. \]

In order to see that the sign of the Lagrange multiplier \( \lambda \) is positive, note that with \((IR_h)\) and \((IR_l)\) binding \((IC_\pi)\) rewrites as
\[
\pi[q_{l+} + \omega Q_{l+i}] - (1 - \pi)q_{l-} - \omega Q_{l-h}] \Delta \leq 0.
\]

Hence, the constraint is relaxed when the RHS rises. Consequently, the shadow cost of the constraint is positive, implying \( \lambda > 0 \).

Consequently, the solution exhibits
\[
q_{l+} = q_{l-} < q^*_l < q_{h+} = q_{h-} = q^*_h Q_{l+i} < Q_{h-l} = Q^*_i < Q_{h+h} = Q^*_h < Q_{l-h}.
\]

Finally, we check that for this solution \((IC_l)\) is satisfied, which is the case if
\[
0 \geq -\pi[q^*_h + \omega Q^*_i] - (1 - \pi)[q^*_l - \omega Q^*_l]
\]
which simplifies to \( \pi \geq \pi^*_i \) and holds because of \( \pi^*_h > \pi^*_i \) and the proposition’s assumption \( \pi > \pi^*_i \).
Q.E.D

**Proof of Proposition 5:** If \( \pi \leq \pi^*_i \), then, necessarily, \( \pi^l \geq 0 \), implying \( \omega \geq \omega^* \equiv q^*_h / Q^*_i \). Note that \( \pi^*_i = 0 \) for \( \omega = \omega^*_i \equiv q^*_h / Q^*_i \). Because \( q^*_i < q^*_l \) and \( Q^*_h > Q^*_i \), we have \( \omega^*_i < \omega^* \). Since \( \pi^*_i \) is increasing in \( \omega \), we therefore have \( \pi^*_i > 0 \) for \( \omega = \omega^* \).

We next show that for \( \omega \geq \omega^* \), we have \( \pi^l < \pi^*_i \). To see this, define
\[
\Delta^l(\omega) \equiv \pi^*_i - \pi^l = \frac{q^*_l + \omega Q^*_i}{\omega Q^*_h + Q^*_i} - \frac{q^*_h + \omega Q^*_i}{\omega Q^*_h + Q^*_i}
\]
so that
\[
\frac{\partial \Delta^l}{\partial \omega} = \frac{q^*_h Q^*_i + Q^*_h Q^*_i - q^*_h Q^*_i Q^*_h}{\omega^2(Q^*_h + Q^*_i)(Q^*_h + Q^*_i)}.
\]

Hence, the sign of \( \partial \Delta^l / \partial \omega \) does not depend on \( \omega \) and is either positive or negative. Recall that at \( \omega = \omega^* \) we have \( \pi^l = 0 \) and \( \pi^*_i > 0 \) so that \( \Delta^l(\omega) = \pi^*_i - 0 > 0 \). Hence, if the sign of \( \partial \Delta^l / \partial \omega \) is positive, it follows \( \Delta^l(\omega) > 0 \) for all \( \omega \geq \omega^* \) and, hence, \( \pi^l < \pi^*_i \). If, on the other hand, the sign of \( \partial \Delta^l / \partial \omega \) is negative, \( \pi^l < \pi^*_i \) results from
\[
\Delta^l(\omega) \geq \lim_{\omega \to \infty} \Delta^l(\omega) = \frac{Q^*_h Q^*_i - Q^*_h Q^*_i}{(Q^*_h + Q^*_i)(Q^*_h + Q^*_i)} = \frac{Q^*_h Q^*_i - Q^*_h Q^*_i}{(Q^*_h + Q^*_i)(Q^*_h + Q^*_i)} > 0,
\]
where the last inequality follows from \( Q^*_h > Q^*_i > 0 \) and \( Q^*_i > Q^*_l > 0 \). We therefore conclude that \( \pi \leq \pi^l \) implies \( \pi \leq \pi^*_i \).
We next show that for \( \pi \leq \pi^l \), the profit maximizing mechanism has \((IC_i)\) and \((IR_h)\) binding, while \((IC_h)\) and \((IR_l)\) are automatically satisfied. That is, it exhibits

\[
U_h = 0; U_l = -\pi[q_{h+} + \omega Q_{h+h}]\Delta - (1 - \pi)[q_{h-} - \omega Q_{h-l}]\Delta
\]

Indeed, substituting these values into \( \Pi \) leads to maximizing

\[
\nu[\pi[((h + \Delta/\tilde{v})q_{h+} - c(q_{h+}))) + \omega((h + \Delta/\tilde{v})Q_{h+h} - C(Q_{h+h}))]
\]

+ \(1 - \pi)[((h + \Delta/\tilde{v})q_{h-} - c(q_{h-})) + \omega((l - \Delta/\tilde{v})Q_{h-l} - C(Q_{h-l}))]
\]

+ \(1 - \nu)[\pi[(lq_{l+} - c(q_{l+})) + \omega(lQ_{l+} - C(Q_{l+})]) + (1 - \pi)[(lq_{l-} - c(q_{l-})) + \omega(hQ_{l-h} - C(Q_{l-h})]}
\]

with optimal values

\[
q_{l+} = q_{l-} = q^*_l < q^*_h < q_{h+} = q_{h-} = q^*_l < Q_{l+} = Q_{l+} = Q_{l-} = Q^*_l = Q^*_h < Q_{h+h} = Q^*_h;
\]

It remains to be checked whether this solution satisfies \((IC_h)\) and \((IR_l)\).

To see that \((IC_h)\) is satisfied at this solution, note that the constraint for this solution simplifies to

\[
(q^*_h - q^*_l + \omega(Q^*_l - Q^*_l)) \geq (1 - \pi)\omega(Q^*_l + Q^*_l - Q^*_l - Q^*_l).
\]

Note that the inequality holds for any \( \pi \) if it holds for \( \pi = 0 \). In this case, the inequality is equivalent to

\[
(q^*_h - q^*_l) \geq \omega(Q^*_l - Q^*_l),
\]

which holds since the left hand side is positive, whereas the right hand side is negative.

Next consider \((IR_l)\) for the solution, stating that

\[
U_l = -\pi[q^*_h + \omega Q^*_h]\Delta - (1 - \pi)[q^*_h - \omega Q^*_l]\Delta = [(1 - \pi)\omega(Q^*_h + Q^*_l) - (q^*_h + \omega Q^*_l)\] \(\Delta
\]

is non-negative, which is a condition that rewrites as

\[
\pi \leq 1 - \frac{q^*_h + \omega Q^*_h}{\omega(Q^*_l + Q^*_l)} = \pi^l.
\]

Q.E.D.

**Proof of Proposition** \([6]\): Assuming the three binding constraints are \((IR_l)\), \((IR_h)\) and \((IC_i)\), we have

\[
U_l = 0; U_h = 0; \pi[q_{h+} + \omega Q_{h+h}]\Delta + (1 - \pi)[q_{h-} - \omega Q_{h-l}]\Delta = 0
\]

Substitution of \( U_h = U_l = 0 \) implies to maximize

\[
W_3 \text{ s.t. } v[\pi[q_{h+} + \omega Q_{h+h}] + (1 - \pi)[q_{h-} - \omega Q_{h-l}] = 0,
\]

with \( W_3 \) as defined in the proof of Proposition \([4]\)
The associated Lagrangian is
\[
\mathcal{L} \equiv \nu[\pi(h q_{h^+} - c(q_{h^+})) + \omega(h Q_{h^+} - C(Q_{h^+}))] + (1 - \pi)[(h q_{h^-} - c(q_{h^-})) + \omega(h Q_{l^-} - C(Q_{l^-}))]
\]
\[
+ (1 - \nu)[\pi(l q_{l^+} - c(q_{l^+})) + \omega(l Q_{l^+} - C(Q_{l^+}))] + (1 - \pi)[(l q_{l^-} - c(q_{l^-})) + \omega(h Q_{l^-} - C(Q_{l^-}))]
\]
\[
- \lambda(\nu[\pi[q_{h^+} + \omega Q_{h^+}] + (1 - \pi)[q_{h^-} - \omega Q_{h^-}]]),
\]
where \( \lambda \) is the Lagrange multiplier. Hence, the optimality conditions w.r.t. \( q_{l^+}, Q_{l^+}, q_{l^-}, Q_{l^-} \) imply \( q_{l^+} = Q_{l^+} = q_{l^-} = q_{l}^* \), and \( Q_{l^-} = q_h^* \); the optimality conditions w.r.t. \( q_{h^+}, Q_{h^+}, q_{h^-} \) coincide; they satisfy
\[
c'(q_{h^+}) = c'(Q_{h^+}) = c'(q_{h^-}) = (h - \lambda)^+;
\]
where for \( Q_{h^-} \) it optimally holds
\[
c'(Q_{h^-}) = (1 + \lambda)^+.
\]

In order to see that the sign of the Lagrange multiplier \( \lambda \) is negative, note that with \((IR_h)\) and \((IR_l)\) binding \((IC_l)\) rewrites as
\[
\pi[q_{h^+} + \omega Q_{h^+}]\Delta + (1 - \pi)[q_{h^-} - \omega Q_{h^-}]\Delta \geq 0
\]
Hence, the constraint is strengthened when the RHS rises. Consequently, the shadow cost of the constraint is negative as raising the right hand side lowers the objective. Hence, \( \lambda < 0 \).

As a result, the solution exhibits
\[
q_{l^+} = q_{l^-} = q_l^* < q_{h^+} = q_h^-; Q_{l^-} < Q_{l^+} = Q_l^* < Q_{h^-} = Q_{h^+}^* < Q_{h^+}.
\]

Finally, we check that for this solution \((IC_h)\) is satisfied, which is the case if
\[
0 \geq \pi[q_{l^+} + \omega Q_{l^+}] + (1 - \pi)[q_{l^-} - \omega Q_{l^-}],
\]
which simplifies to \( \pi \leq \pi_{l^+}^* \) and holds because of \( \pi_{l^+} < \pi_{l}^* \) and the proposition’s assumption \( \pi < \pi_{l}^* \).

Q.E.D

**Proof of Proposition** [7] First, consider \( \pi \in [0, \pi_{l^+}^*] \). In this case, \( \omega > \omega_{l^+}^* \) so that \( \pi < \pi_{l^+}^* \) for \( \omega > \omega_{l^+}^* \).

As a result, the optimal contract is characterized by either Proposition [5] or [6]. In the first case, we have
\[
Q_{h^-} = Q_{l^+}^* < Q_{l^+} = Q_{l^-}^* < Q_{l^-} = Q_{h^-}^* < Q_{h^+} = Q_{h^+}^*.
\]

so that the result follows directly. In the second case, we have for any \( \omega > \omega_{l^+}^* \) that
\[
Q_{h^-} < Q_{l^+} = Q_{l^-}^* < Q_{l^-} < Q_{h^-}^* < Q_{h^+}^* < Q_{h^+}.
\]

so that the result follows when \( Q_{h^-} \) is decreasing and \( Q_{h^+} \) is increasing in \( \omega \). To see that this is indeed the case, note that at the solution the constraint
\[
\pi q_{h^+} + (1 - \pi)q_{h^-} + \omega(\pi Q_{h^+} - (1 - \pi)Q_{h^-}) \geq 0
\]
binds so that the Lagrange multiplier is strictly negative. Moreover, it holds \( \pi Q_{h+l} - (1 - \pi)Q_{h-l} < 0 \) so that the constraint tightens as \( \omega \) grows, meaning that the Lagrange multiplier \( \lambda \) becomes more positive. As a result, the upward distortion on \( Q_{h+l} \) and the downward distortion on \( Q_{h-l} \) intensify as \( \omega \) rises. This confirms that \( Q_{h-l} \) as derived in Proposition 5 is decreasing and \( Q_{h+l} \) as derived in Proposition 6 is increasing in \( \omega \) and the results follows.

Second, consider \( \pi \in (\bar{\pi}_l^*, \bar{\pi}_h^*]. \) In this case, \( \omega_l^* < 0 \) and \( \omega_h^* \geq 0 \) so that \( \pi \in (\bar{\pi}_l^*, \bar{\pi}_h^*] \) for \( \omega > \omega_h^* \). As a result, the optimal contract is characterized by Proposition 2, which yields the result.

Finally, consider \( \pi \in (\bar{\pi}_h^*, 1]. \) In this case, \( \omega_l^*, \omega_h^* < 0 \) so that \( \pi > \bar{\pi}_h^* \) for \( \omega > 0 \). As a result, the optimal contract is characterized by either Proposition 3 or 4. In the first case, we have

\[
Q_{l+1} = Q_l^h < Q_{h-l} = Q_l^i < Q_{h+h} = Q_h^* < Q_{l-h} = Q_l^h
\]

so that the result follows directly. In the second case, we have for any \( \omega > 0 \) that

\[
Q_{l+1} < Q_{h-l} = Q_l^i < Q_{h+h} = Q_h^* < Q_{l-h}
\]

so that the result follows when \( Q_{l+1} \) is decreasing and \( Q_{l-h} \) is increasing in \( \omega \). To see that this is indeed the case, note that at the solution the constraint

\[
\pi q_{l+} + (1 - \pi) q_{l-} + \omega(\pi Q_{l+1} - (1 - \pi)Q_{l-h}) \leq 0
\]

binds. Hence, the associated Lagrange multiplier is strictly positive and, moreover, it holds \( \pi Q_{l+1} - (1 - \pi)Q_{l-h} < 0 \). The constraint therefore tightens as \( \omega \) grows, meaning that the Lagrange multiplier \( \lambda \) becomes more positive. Consequently, the upward distortion on \( Q_{l-h} \) and the downward distortion on \( Q_{l+1} \) intensify as \( \omega \) rises. This confirms that \( Q_{l+1} \) as derived in Proposition 4 is decreasing and \( Q_{l-h} \) as derived in Proposition 4 is increasing in \( \omega \) so that the result follows.

**Q.E.D.**

**Proof of Lemma 2.** The first observation follows directly from the definition of \( \bar{\pi}_l^* \) and \( \bar{\pi}_h^* \). Moreover, using the implicit definitions of \( q_l^h, Q_l^h \), it follows that these quantities are positive if and only if \( \nu < \bar{\nu}_h \equiv \bar{l}/h \). Hence, \( \bar{\pi}_l^* < 1 \) and only if \( \nu < \bar{\nu}_h \). For \( \nu = 0 \), we have that \( q_l^h, Q_l^h \), and \( Q_l^i \) are efficient so that \( \bar{\pi}_l^* = \pi_l^* \). Furthermore, it follows

\[
\frac{d\bar{\pi}_l^*}{d\nu} = \frac{\partial \bar{\pi}_l^*}{\partial q_l^h} \frac{\partial q_l^h}{\partial v} + \frac{\partial \bar{\pi}_l^*}{\partial Q_l^h} \frac{\partial Q_l^h}{\partial v} + \frac{\partial \bar{\pi}_l^*}{\partial q_l^i} \frac{\partial q_l^i}{\partial v} + \frac{\partial \bar{\pi}_l^*}{\partial Q_l^i} \frac{\partial Q_l^i}{\partial v} \geq 0,
\]

where \( \partial \bar{\pi}_l^* / \partial Q_l^h \leq 0 \) follows from \( \bar{\pi}_l^* > 0 \).

Likewise, using the implicit definitions of \( q_h^l, Q_h^l \), and \( Q_l^i \), that \( \bar{\pi}_l^* \) is increasing, whenever \( \bar{\pi}_l^* > 0 \) follows from

\[
\frac{d\bar{\pi}_l^*}{d\nu} = \frac{\partial \bar{\pi}_l^*}{\partial q_h^l} \frac{\partial q_h^l}{\partial v} + \frac{\partial \bar{\pi}_l^*}{\partial Q_h^l} \frac{\partial Q_h^l}{\partial v} + \frac{\partial \bar{\pi}_l^*}{\partial q_l^i} \frac{\partial q_l^i}{\partial v} + \frac{\partial \bar{\pi}_l^*}{\partial Q_l^i} \frac{\partial Q_l^i}{\partial v} \geq 0,
\]

where \( \partial \bar{\pi}_l^* / \partial Q_h^l \leq 0 \) follows from \( \bar{\pi}_l^* > 0 \). For \( \nu = 1 \), it follows that the quantities \( q_h^l, Q_h^l \), and \( Q_l^i \) are efficient so that \( \bar{\pi}_l^* = \pi_l^* \). As \( \nu \) approaches 0, \( \phi \) approaches 0, and the quantities \( q_h^l \) and \( Q_h^l \) raise
without bound, whereas $Q_l^i$ becomes 0. This implies that starting from $\nu = 1$, where $\bar{\pi}^i = \bar{\pi}^i_*$, the value $\bar{\pi}^i$ decreases when we lower $\nu$ and there is some cutoff level $\bar{\nu}_l \in (0, 1)$ where $\bar{\pi}^i$ equals 0 and is negative below $\bar{\nu}_l$. Q.E.D.

References


