Efficient trade on networks*

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Abstract

We analyze trade on networks with costly transportation and show that, subject to incentive compatibility, individual rationality, and no-deficit constraints, ex post efficient trade is impossible for any ownership structure if the transportation cost is sufficiently large. For many networks, including the star and the wheel with identical distributions, the first-best—optimal ownership followed by ex post efficient trade—is impossible for any positive transportation costs because the first-best places the resource at the hub, which, by a generalization of the Myerson-Satterthwaite theorem, prevents ex post efficient trade. We derive the constrained-efficient trade mechanism and use it to determine the optimal ownership structure. educing transportation costs has the hitherto unnoticed benefits of making the market work better and of allowing for more efficient ownership structure.

Keywords: mechanism design, networks, partnership models, transaction costs, supply chains
JEL Classification: D44, D82, L41

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1 Introduction

Trade typically occurs along networks and involves costly transportation. For example, the costs of shipping a graphic novel from France to Australia easily outweigh its price. Notwithstanding, much, if not all, of the mechanism design literature to date abstracts from transportation costs and implicitly rests on the assumption that the trading network is complete, that is, that every buyer can, in principle and at zero additional cost, trade with any seller. This raises the question of whether these are innocuous simplifications that allow the analysis to focus on what is essential or whether they stand in the way of relevant insights. In this paper, we argue that it is the latter. Specifically, we show that transportation costs are a tractable modelling device to capture transactions costs above and beyond those that derive from private information and that jointly transportation costs and private information can prevent efficient trade. Our analysis implies that, among other things, reducing transportation (or trade) costs has the hitherto unnoticed benefits of improving the performance of the market and of permitting better initial allocation of resources.

The key insights are readily summarized by considering a bilateral trade setting with independently and continuously distributed private values. For simplicity, assume that the buyer and the seller draw their values from identical distributions. By the Myerson and Satterthwaite theorem, even in the absence of transportation costs, ex post efficient trade is already impossible without running a deficit. So, it is not a big surprise that ex post efficient trade remains impossible if transportation is costly. Intuition gleaned from the partnership literature without transportation costs suggests that with an appropriately chosen ownership structure—say, fifty-fifty—ex post efficient trade becomes possible.[1] As we show, this intuition is correct when the transportation cost is small but not when it is large: if the transportation cost is one-half of the maximally possible gains from trade without transportation costs or larger, then there is no ownership structure that permits ex post efficient trade. That is, incentive costs together with sufficiently large transportation costs imply insurmountable transactions costs—ex post efficient trade is impossible for any ownership structure. Put differently, reducing transportation costs has the additional, and to our knowledge hitherto overlooked, benefit of making markets work better.

Moreover, while in the absence of transportation costs equal ownership is optimal anticipating ex post efficient trade with identical distributions, the uniquely optimal ownership

[1] See, for example, Cramton et al. (1987).

structure anticipating ex post efficient trade is extremal under natural generalizations, including star networks with three or more agents or heterogeneous distributions. But, as we show in a generalization of the Myerson-Satterthwaite theorem to networks with costly transportation, with extremal ownership ex post efficient trade is impossible for *any* network. Beyond making markets work better, reducing transportation costs has yet another benefit—it permits a more efficient initial allocation of resources.

Our analysis assumes commonly known, constant marginal costs of transportation that are the same across every link in the network. Under the assumptions of identical distributions that exhibit increasing virtual types, we characterize the second-best (or *constrained-efficient*) trading mechanism for networks when ex post efficient trade is impossible. We illustrate this for the star and the wheel networks, assuming uniform distributions. With the constrained-efficient mechanism in hand, we then solve for the optimal ownership structure, anticipating the use of the optimal trading mechanism for any given ownership structure, and we compare these ownership structures to the first-best ones, which would be obtained if trade were ex post efficient independently of the ownership structure. For the star and wheel network and identical distributions, the first-best ownership structure is extremal and endows the hub with all the resources. Even though this prevents ex post efficient trade, the optimal ownership structure anticipating the use of the constrained-efficient mechanism also endows the hub with all the resources if the marginal cost of transportation is large enough (e.g., larger than one-half of the maximally possible gains from trade without transportation costs). Moreover, the optimal ownership structure is not necessarily monotone in the marginal cost of transportation.

Because the constrained-efficient mechanism depends on the distributions, it is not *detail-free* (or prior-free), which is often seen as a desirable property (Wilson, 1987). With that in mind, we also propose detail-free trade-sacrifice mechanisms for networks that always balance the budget, endow the agents with dominant strategies, respect their individual rationality constraints ex post, and aim to allocate close to efficiently. Extensions allow for costly entry, analyze heterogeneous distributions, and analyze star networks in which there is no agent at the hub.

The paper relates to the literature on the (im)possibility of ex post efficient trade initiated by Vickrey (1961) and Myerson and Satterthwaite (1983) and debates surrounding the Coase Theorem (Coase, 1960). That extremal ownership structure prevents ex post efficient trade follows from an extension of the impossibility theorem of Myerson and Satterthwaite to costly transportation.[2] That ex post efficient trade is not possible for any ownership

[2]Of course, it is also an extension of the bilateral trade setup to settings with multiple buyers and one seller, but that extension is already in the literature (see e.g. Gresik and Satterthwaite, 1989).

structure if the marginal cost of transportation is sufficiently large is, to our knowledge, a new impossibility theorem. The fact that, if the marginal cost of transportation is small, the designer trades off incentive costs against transportation costs builds on the insight from the partnership literature that, without costly transportation, ex post efficient trade is possible with appropriately structured ownership; see, for example, Cramton et al. (1987), Che (2006), or Figueroa and Skreta (2012). We show that this insight extends to costly transportation, provided the marginal cost of transportation is sufficiently small. To solve the designer's problem, the paper builds on the work related to optimal trading mechanisms for asset markets—problems in which each agent's trading positions (buy, sell, remain inactive) are determined endogenously—and partnership models by Lu and Robert (2001) and Loertscher and Wasser (2019).[3] Our trade-sacrifice mechanism for networks, which in contrast to the constrained-efficient mechanism is prior-free, is related to the earlier work by Hagerty and Rogerson (1987), McAfee (1992) and Loertscher and Marx (2020b). There is a related literature on networks, including Akbarpour and Jackson (2018), which examines how diffusion patterns depend on the network placement of heterogeneous agents. In contrast, we examine how trade patterns depend on the ownership (or placement) of resources, holding fixed the network locations of the agents.

The remainder of this paper is structured as follows. Section 2 contains the setup together with the definitions of the various problems of interest and basic results. In Section 3, we provide possibility and impossibility results for efficient trade on networks, characterize the constrained-efficient mechanism, and define a detail-free trade-sacrifice mechanism for networks. In Section 4, we consider the optimal ownership structure. Extensions are presented in Section 5, and Section 6 concludes the paper.

2 Setup

We assume n agents indexed by $i \in \mathcal{N} \equiv \{1, \ldots, n\}$ and a resource whose total supply is 1. Each agent $i \in \mathcal{N}$ is located at a node in an undirected graph that connects all agents, where $d_{ij} \in \{0, 1, \ldots, n-1\}$ is the length of the minimum path through the network between agents i and j (for all $i \in \mathcal{N}$, $d_{ii} = 0$). Initially resource amount r_i is held by agent i, where $r_i \in [0, 1]$ and $\sum_{i \in \mathcal{N}} r_i = 1$. We refer to the vector $\mathbf{r}$ as the ownership structure. Each agent i has constant marginal value v_i for the resource, which is independently drawn from the continuous distribution F_i with support $[0, 1]$ and density $f_i > 0$. For some of our results,

[3] The term "asset market" has been used by Loertscher and Marx (2020b, 2023) and Delacrétaz et al. (2022). Analyses of asset market problems (that do not use that label) are also provided by Lu and Robert (2001) and Li and Dworczak (2021).

we assume that all agents draw their values from the same distribution F whose density is denoted f.

For the analysis involving constrained efficient mechanisms (defined below), we assume that each agent i's virtual type functions,

$$\Psi_i^B(v) \equiv v - \frac{1 - F_i(v)}{f_i(v)} \quad \text{and} \quad \Psi_i^S(v) \equiv v + \frac{F_i(v)}{f_i(v)},$$

are increasing. Despite this monotonicity of the virtual types, which corresponds to what Myerson (1981) calls the "regular" case, the mechanism design problem in the trading phase faced by the designer will not be regular away from extremal ownership structure. In the case of identical distributions, we simply write $\Psi^B(v)$ and $\Psi^S(v)$.

We assume that the cost of transporting x units of the resource from agent i to agent j is xcd_{ij}, where $c \geq 0$ is the commonly known marginal transportation cost per edge traveled. The $n \times n$ symmetric matrix $C = (C_{ij})_{i,j \in \mathcal{N}}$ is called a transportation cost matrix, with component C_{ij} representing the transportation cost between agents i and j, if for all $i, j \in \mathcal{N}$, $C_{ii} = 0$ and $C_{ij} = C_{ji} = cd_{ij}$.

Two agents i and j are directly connected if $d_{ij} = 1$. We say that a network is ***complete*** if every agent is directly connected to every other agent. Agent i is said to be ***completely connected*** if agent i is directly connected to every other agent, and we say that agent i is ***maximally connected*** if no other agent is directly connected to a larger number of other agents than is agent i. For example, the ***star*** network with $n \geq 3$ agents is defined as having agent 1 as the hub and a transportation cost matrix such that for $i > 1$, $C_{1i} = c$, and for $1 < i < j$, $C_{ij} = 2c$. We define a ***wheel*** network with $n \geq 5$ agents to also have agent 1 as the hub, but with a transportation cost between two agents i and j with $i < j$ of $C_{ij} = c$ if either (i) $i = 1$ or (ii) $i > 1$ and $j - i \in \{1, n-2\}$, and otherwise a transportation cost of $C_{ij} = 2c$.

Trading mechanisms

A trading mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ consists of allocation rule $\mathbf{Q} = (Q_i)_{i \in \mathcal{N}}$, where $Q_i(\mathbf{v})$ specifies agent i's quantity following trade, and payment rule $\mathbf{M}$, where $M_i(\mathbf{v})$ specifies the payment made by agent i to the mechanism. For the allocation rule, we have $Q_i : [0,1]^n \to [0,1]$ such that $\sum_{i \in \mathcal{N}} Q_i(\mathbf{v}) = 1$. For the payment rule, for $i \in \mathcal{N}$, we have $M_i : [0,1]^n \to \mathcal{R}$.

Ex post efficient trade

Given realized types $\mathbf{v}$ and transportation cost matrix C, define the $n \times n$ binary matrix $\hat{V}(\mathbf{v}, C)$ each of whose rows sums to 1 by:[4]

$$\hat{V}_{ij}(\mathbf{v}, C) = \begin{cases} 1 & \text{if } v_j - C_{ij} \geq \max_\ell v_\ell - C_{i\ell} \text{ and } v_j - C_{ij} > \max_{\ell<j} v_\ell - C_{i\ell}, \\ 0 & \text{otherwise.} \end{cases}$$

This says that $\hat{V}_{ij}(\mathbf{v}, C) = 1$ only if allocating agent i's resources to agent j maximizes value net of transportation costs. If there are ties, then we (arbitrarily) identify the lowest indexed agent j with whom the maximum net value is achieved.

The ex post efficient allocation rule assigns to agent i the resources of agents j with $\hat{V}_{ji}(\mathbf{v}, C) = 1$:

$$Q^e_{i,\mathbf{r},C}(\mathbf{v}) \equiv \sum_{j \in \mathcal{N}} \hat{V}_{ji}(\mathbf{v}, C) r_j.$$

Maximized social surplus is

$$SS^e_{\mathbf{r},C}(\mathbf{v}) = \sum_{i=1}^{n} \sum_{j=1}^{n} (v_i - C_{ji})\, \hat{V}_{ji}(\mathbf{v}, C) r_j, \tag{1}$$

whose expected value is

$$ss^e_C(\mathbf{r}) \equiv \mathbb{E}_{\mathbf{v}}[SS^e_{\mathbf{r},C}(\mathbf{v})]. \tag{2}$$

Total transportation costs under the ex post efficient allocation rule are:

$$T^e_{\mathbf{r},C}(\mathbf{v}) \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ji} \hat{V}_{ji}(\mathbf{v}, C) r_j,$$

whose expected value is

$$t^e_{\mathbf{r},C} \equiv \mathbb{E}_{\mathbf{v}}[T^e_{\mathbf{r},C}(\mathbf{v})]. \tag{3}$$

First-best

The linearity in $\mathbf{r}$ of social surplus under ex post efficient trade, $SS^e_{\mathbf{r},C}(\mathbf{v})$ defined in (1), implies that its expectation, $ss^e_C(\mathbf{r})$ defined in (2), is also linear in $\mathbf{r}$. This in turn implies that an extremal ownership structure, i.e., $r_i = 1$ for some $i \in \mathcal{N}$, is always optimal.[5] We

[4]This specification of $\hat{V}$ uses a particular tie-breaking rule, but given our assumptions ties are zero probability events and so the particular tie-breaking rule does not affect our results.

[5]Linearity of $SS^e_{\mathbf{r},C}(\mathbf{v})$ in $\mathbf{r}$ means the for any $\mathbf{r}, \mathbf{r}'$ and any $\alpha \in [0, 1]$, we have $SS^e_{\alpha\mathbf{r}+(1-\alpha)\mathbf{r}',C}(\mathbf{v}) = \alpha SS^e_{\mathbf{r},C}(\mathbf{v}) + (1-\alpha) SS^e_{\mathbf{r}',C}(\mathbf{v})$, which is evidently the case.

state this in the following proposition, where we consider the case of a central authority in charge of the resource placement and its subsequent trade, which we refer as the "planner's problem":[6]

Proposition 1. *The planner's problem has a solution involving an extremal ownership structure followed by ex post e cient trade according to* $\mathbf{Q}^e_{\mathbf{r},C}(\mathbf{v})$*.*

Proof. See Appendix A.

While an extremal ownership structure is not necessarily uniquely optimal, and not any extremal ownership structure will be optimal, one always will be. Specifically, given agent $j \in \{1,\ldots,n-1\}$ with $\frac{\partial ss^e_C(\mathbf{r})}{\partial r_j} = \max_{\ell \in \{1,\ldots,n-1\}} \frac{\partial ss^e_C(\mathbf{r})}{\partial r_\ell} \geq 0$, the extremal ownership structure that has $r_j = 1$ is optimal, and if $\frac{\partial ss^e_C(\mathbf{r})}{\partial r_j} < 0$ for all $j \in \{1,\ldots,n-1\}$, then the extremal ownership structure that has $r_n = 1$ is uniquely optimal. If $\frac{\partial ss^e_C(\mathbf{r})}{\partial r_j} = 0$ for all $j \in \{1,\ldots,n-1\}$, as is the case, for example, if $c = 0$, then any $\mathbf{r}$ is optimal.

While Proposition 1 shows that an extremal ownership structure is optimal in the planner's problem, for some network structures we can also identify with which agent or agents the resources should be placed. For networks with at least one completely connected agent and $F_i = F$ for all $i \in \mathcal{N}$, we have the following result:

Proposition 2. *If the set* $\mathcal{I} \subset \mathcal{N}$ *of completely connected agents is nonempty and* $F_i = F$ *for all* $i \in \mathcal{N}$*, then* $r_i = 1$ *for any* $i \in \mathcal{I}$*, followed by ex post e cient trade according to* $Q^e_{\mathbf{r},C}(\mathbf{v})$*, solves the planner's problem; moreover, for* $c \in (0,1)$*, no* $\mathbf{r}$ *with* $r_j > 0$ *for* $j \in \mathcal{N} \setminus \mathcal{I}$ *solves the planner's problem.*

Proof. See Appendix A.

Applying Proposition 2 to the star and wheel networks, which have a unique completely connected agent, referred to as the "hub," we have the following corollary:

Corollary 1. *For star and wheel networks with* $F_i = F$ *for all* $i \in \mathcal{N}$*, if* $c \in (0,1)$*, then the unique solution to the planner's problem places all resources at the hub, followed by ex post e cient trade.*

Walrasian prices to implement ex post efficiency

Given a network, we refer to a set of agents that, under ex post efficiency, trade only among themselves as a *cluster*. Each cluster will have one agent whose allocation increases relative to

[6]While we assume identical distributions for tractability and to focus attention on the optimal ownership structure in the presence of transportation and incentive costs on a network, the proof of Proposition 1 makes clear that the result generalizes straightforwardly to any independent type distributions.

its initial resources, which we refer to as the buyer, and one or more agents whose allocations decrease relative to their initial resources, which we refer to as the sellers. We work with the convention that the sellers need to cover the transportation cost. For a star network, there can only be one buyer under ex post efficiency and so only one cluster. In that case, any price between the highest seller value and the buyer's value will be a Walrasian (market-clearing) price, and no other price will be.

For other networks, such as a wheel network, there can be multiple clusters. In that case, Walrasian prices must still satisfy necessary conditions within each cluster. But, in addition, the Walrasian prices must prevent sellers from one cluster from having an incentive to sell to the buyer in another cluster.

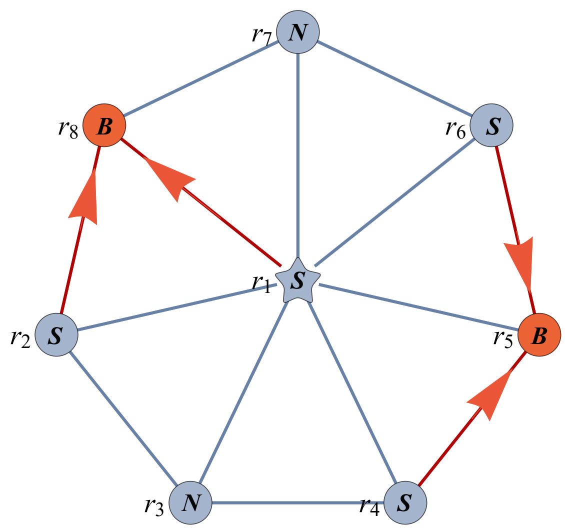

Figure 1: Illustration of an ex post efficient outcome with two trading clusters for a wheel network with $n = 8$. Agents that are buyers are labeled with B, sellers are labeled with S, and nontrading agents are labeled with N.

In the illustration in Figure 1of a wheel network with $n = 8$ that has an ex post efficient outcome that involves two trading clusters, it follows that:[7]

$$\max\{v_1, v_2\} < v_8 - c < v_5 < v_8 \tag{4}$$

and:[8]

$$\max\{v_4, v_6\} < v_5 - c. \tag{5}$$

[7]Because agent 1's resources are allocated to agent 8, it follows that $v_8 = \max \mathbf{v} \geq v_5$ and $v_8 > v_1 + c$. Because agent 2's resources are allocated to agent 8, we have $v_8 > v_2 + c$. Because the resources of agents 4 and 6 are allocated to agent 5 instead of to agent 8, we have $v_8 - 2c < v_5 - c$.

[8]Because agent 4's and agent 6's resources are allocated to agent 5, it follows that $v_5 > v_4 + c$ and $v_5 > v_6 + c$.

We refer to the cluster involving agents 1-2-8 as cluster 1 and the cluster involving agents 4-5-6 as cluster 2. Focusing on cluster 1, the minimum and maximum Walrasian prices are $\underline{p}_1^w \equiv \max\{v_1, v_2\} + c$ and $\overline{p}_1^w \equiv v_8$. Focusing on cluster 2, the minimum and maximum Walrasian prices are $\underline{p}_2^w \equiv \max\{v_4, v_6\} + c$ and $\overline{p}_2^w \equiv v_5$. By (4) and (5),

$$\underline{p}_1^w < \overline{p}_1^w \quad \text{and} \quad \underline{p}_2^w < \overline{p}_2^w.$$

In addition to the above constraints, Walrasian prices for the broader economy must leave agents with no incentive to sell to a different cluster. To ensure that the agents identified as sellers in Figure 1 have no incentive to sell to other agents, it is sufficient that agent 1 has no such incentive, so we require that Walrasian prices p_1^w and p_2^w satisfy $p_2^w \leq p_1^w$. So that the buyers in Figure 1 do not have an incentive to sell to the buyer in the other cluster, it is sufficient that agent 5 does not prefer to sell to agent 8, so we require that $p_1^w - c \leq p_2^w$. Putting these together, we have

$$p_1^w - c \leq p_2^w \leq p_1^w. \tag{6}$$

Thus, the ex post efficient outcome shown in Figure 1 can be supported as a Walrasian equilibrium with any prices in

$$W \equiv \left\{ (p_1, p_2) \in [\underline{p}_1^w, \overline{p}_1^w] \times [\underline{p}_2^w, \overline{p}_2^w] \mid p_1 - c \leq p_2 \leq p_1 \right\}.$$

Such prices exist as long as $\underline{p}_1^w - c < \overline{p}_2^w$, which we can write as $\max\{v_1, v_2\} < v_5$, which holds by (4), implying that W is nonempty.

As this shows, the set of Walrasian prices in the whole economy is a restricted set of the Walrasian prices in the standalone clusters, where we require prices in different clusters to remain sufficiently close to one another. In the limit as c goes to zero, prices must be equal across clusters.

3 Trade on networks

We now analyze trade on networks. We first provide possibility and impossibility results for ex post efficient trade. Second, we derive the constrained efficient mechanism when ex post efficient trade is not possible and characterize conditions under which only constrained efficient trade is possible.

3.1 Impossibility of ex post efficient trade

We now show that a generalization of the Myerson-Satterthwaite theorem (Myerson and Satterthwaite, 1983) holds if the ownership structure is extremal. An extremal ownership structure is one in which one agent is the sole owner of the resource, who is therefore the seller whenever there is trade. We take this agent to be agent 1 and denote that agent's type by θ and by $\tilde{\mathbf{v}}$ the values of the $n-1$ other agents, who will only ever trade as buyers.

Trade between the seller and buyer i is ex post efficient if and only if $\tilde{v}_i - C_{1i} = \max_{j\in\{2,\ldots,n\}} \tilde{v}_j - C_{1j}$ and

$$\tilde{v}_i - C_{1i} > \theta.$$

Consider then the market-clearing (Walrasian) prices that establish ex post efficient trade given θ and $\tilde{\mathbf{v}}$. Without loss of generality, we let the buyer bear the transportation cost, so if p^W is a Walrasian price, then it has to satisfy

$$\underline{p}^W \equiv \max\{\theta, \underset{j\in\{2,\ldots,n\}}{\text{2nd}} \tilde{v}_j - C_{1j}\} \le p^W \le \max_{j\in\{2,\ldots,n\}} \tilde{v}_j - C_{1j} \equiv \overline{p}^W,$$

where 2nd selects the second-highest element of a set, and where we define $\underset{j\in\{2,\ldots,n\}}{\text{2nd}} \tilde{v}_j - C_{1j}$ to be zero if $n = 2$. The first inequality ensures that the seller is willing to sell and that the buyer with the second-highest value does not want to buy, while the second inequality ensures that the buyer with the highest value is willing to buy. Consequently, the lowest Walrasian price is $\underline{p}^W$ and the highest Walrasian price is $\overline{p}^W$. As is reasonably well known and easily established, a trading buyer's payment in the VCG mechanism is $\underline{p}^W$ and a trading seller's payment is $\overline{p}^W$ (see e.g. Delacrétaz et al., 2022).[9] Consequently, if trade occurs under ex post efficiency, then the revenue of the mechanism is

$$\underline{p}^W - \overline{p}^W \le 0,$$

where the inequality is strict unless $\text{2nd}_{j\in\{2,\ldots,n\}}\tilde{v}_j - C_{1j} = \max_{j\in\{2,\ldots,n\}} \tilde{v}_j - C_{1j}$. Because ties have probability 0 with continuous distributions, it follows that the VCG mechanism almost always runs a deficit when trade is ex post efficient (and never a budget surplus). Consequently, in expectation, the VCG mechanism runs a deficit. Because the ex post and hence

[9] Social welfare, defined as the gains from trade, with the buyer with the maximum net value present excluding its value for the allocation, is $-\theta$, whereas social welfare with that buyer reporting a value of 0 is $\max\{0, \text{2nd}_{j\in\{2,\ldots,n\}}\tilde{v}_j - C_{1j} - \theta\}$. Hence, the VCG transfer of a trading buyer with the highest net value is $\max\{0, \text{2nd}_{j\in\{2,\ldots,n\}}\tilde{v}_j - C_{1j} - \theta\} - (-\theta) = \max\{\theta, \text{2nd}_{j\in\{2,\ldots,n\}}\tilde{v}_j - C_{1j}\} = \underline{p}^W$. Similarly, social welfare with the seller present but without its value for the allocation, is $\max_{j\in\{2,\ldots,n\}} \tilde{v}_j - C_{1j}$ whereas social welfare is 0 without the seller present or with the seller reporting a cost of 1. Hence, the VCG payment that a trading seller receives is $\max_{j\in\{2,\ldots,n\}} \tilde{v}_j - C_{1j} = \overline{p}^W$.

interim expected payoffs are zero for buyers of type 0 and for the seller of type 1, it follows that the VCG mechanism satisfies the interim individual rationality constraints with equality. By the payoff equivalence theorem, this implies that no other ex post efficient, (Bayesian or dominant strategy) incentive compatible, and interim individually rational mechanism runs a smaller deficit. Because the VCG mechanism runs a deficit, it follows that ex post efficiency is impossible for any network when ownership is extremal. We summarize this in the following result:

Proposition 3. *If $r_i = 1$ and $C_{ij} < 1$ for some $j \in \mathcal{N} \backslash \{i\}$, then ex post e cient trade is impossible.*

If $C_{1i} \geq 1$ for all $i \neq 1$, then trade is never ex post efficient, and ex post efficient trade is possible in the same trivial way that it would be possible in Myerson and Satterthwaite (1983) if the upper bound of the support of the buyer's value distribution were less than the lower bound of the support of the seller's cost distribution.

Corollary 1 and Proposition 3 imply immediately:

Corollary 2. *For star and wheel networks with $F_i = F$ for all $i \in \mathcal{N}$ and $c \in (0, 1)$, the first-best—optimal resource placement followed by ex post e cient trade—is not possible.*

Given Proposition 3 (and Corollary 2), it is clear that we will need to consider constrained efficient trade if $r_i = 1$ for some $i \in \mathcal{N}$. For the case of $c \geq 1/2$, we can already deduce that we will require the use of only constrained efficient trade because, as our next result shows, ex post efficient trade is impossible for *any* $\mathbf{r}$ if $c \geq 1/2$.[10] As intuition for the result, note that under ex post efficiency, any agent of type $v \leq 1 - c$ ever only trades as a seller, and any agent of type $v \geq c$ ever only trades as a buyer. Consequently, for $c \geq 1/2$, agents with types $v \in [1 - c, c]$ never trade and have payoffs of 0. As a result, for $c \geq 1/2$, the trading problem is not only ex post two-sided but already *ad interim*—knowing only its type, every agent knows whether it will trade as a buyer (if $v > c$) or as a seller (if $v < 1 - c$) if it trades and agents with types between $1 - c$ and c know that they will never trade. As in the proof of Proposition 3, it therefore suffices to verify that transportation costs are not covered under VCG transfers, and that the VCG mechanism satisfies the agents' ex post individual rationality constraints with equality. This then also means that it satisfies interim individual rationality with equality.

Proposition 4. *For $c \geq 1/2$, ex post e cient trade is impossible for any network and any ownership structure.*

[10]This uses our assumption that the support of the agents' type distribution is $[0, 1]$. For a more general support of $[\underline{v}, \overline{v}]$, the required condition on transportation costs is that $c \geq (\overline{v} + \underline{v})/2$.

Proof. See Appendix A.

Proposition 4 provides a simple sufficient condition for ex post efficient trade to be impossible.

3.2 Possibility of ex post efficient trade

We now provide a necessary and sufficient condition for ex post efficient trade to be possible. To do so, it will be useful to begin with two lemmas, which characterize, for any incentive compatible mechanism, agents' worst-off types and expected payments to the mechanism.

Consider an incentive compatible mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ and let $u_i(v) \equiv q_i(v)v - m_i(v) - r_i v$ denote the interim expected gains from participation in the mechanism, net of its outside option, of agent i. Incentive compatibility implies that q_i is nondecreasing, from which it follows that the first-order condition $u_i'(v) = q_i(v) - r_i = 0$ characterizes a global minimum for agent i interim expected payoff, provided that it is satisfied for some v. The following lemma, a version of which was first established by Cramton et al. (1987), characterizes the set of worst-off types for any allocation rule such that q_i is non-decreasing:

Lemma 1. *Given an incentive compatible, individually rational mechanism* $\langle \mathbf{Q}, \mathbf{M} \rangle$*, if there is a* v_i *such that* $q_i(v_i) = r_i$*, then the set of worst-o types for agent* i *is* $\{v_i \mid q_i(v_i) = r_i\}$*. If* $q_i(v_i) \neq r_i$ *for all* $v_i \in [0, 1]$*, then the set of worst-o types for agent* i *is the singleton set* $\{v_i \mid q_i(v) < r_i \ \forall v < v_i \text{ and } q_i(v) > r_i \ \forall v > v_i\}$.

As observed by Cramton et al. (1987), intuitively, the worst-off type of an agent expects on average to be neither a net buyer nor a net seller, and therefore an agent with the worst-off type has no incentive to overstate or understate its valuation and so does not need to be compensated to induce truthful reporting, which is why it is the worst-off type.

Given an incentive compatible mechanism, we can use standard mechanism design techniques to write an agent's expected payment to the mechanism in terms of its worst-off type and its virtual type functions. Defining

$$\Psi_i(v; \omega) \equiv \begin{cases} \Psi_i^S(v) & \text{if } v \leq \omega, \\ \Psi_i^B(v) & \text{if } v > \omega, \end{cases}$$

we have:

Lemma 2. *Given an incentive compatible mechanism* $\langle \mathbf{Q}, \mathbf{M} \rangle$ *and agent* i *with initial resources* r_i*, for any* $\omega_i \in [0, 1]$*, agent* i*'s expected payo to the mechanism can be written as*

$$\mathbb{E}_{\mathbf{v}}[M_i(\mathbf{v})] = \mathbb{E}_{\mathbf{v}}\left[\Psi_i(v_i; \omega_i) Q_i(\mathbf{v})\right] - \omega_i r_i - u_i(\omega_i).$$

Proof. See Appendix A.

Letting $\omega^e_{i,\mathbf{r},C}$ denote agent i's worst-off type (or one of its worst-off types) under the ex post efficient allocation rule $Q^e_{i,\mathbf{r},C}$, and using Lemma 2, we obtain an expression for the expected budget surplus of an ex post efficient trading mechanism that satisfies the agents' individual rationality constraints with equality:

$$\Pi^e_{\mathbf{r},C} \equiv \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n}\Psi_i(v_i;\omega^e_{i,\mathbf{r},C})Q^e_{i,\mathbf{r},C}(\mathbf{v})\right] - \sum_{i=1}^{n}\omega^e_{i,\mathbf{r},C}r_i.$$

This now gives us the following result:

Proposition 5. *Ex post e cient trade is possible if and only if* $\Pi^e_{\mathbf{r},C} \geq t^e_{\mathbf{r},C}$.

Given Proposition 5, we can write the necessary and sufficient condition for the possibility of ex post efficient trade as

$$\mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(\Psi_i(v_i;\omega^e_{i,\mathbf{r},C}) - C_{ji})\hat{V}_{ji}(\mathbf{v},C)r_j\right] \geq \sum_{i=1}^{n}\omega^e_{i,\mathbf{r},C}r_i. \tag{7}$$

Condition (7) implicitly defines the set of combinations of ownership structures $\mathbf{r}$ and transportation cost matrices C such that ex post efficient trade is possible. We can use (7) to calculate, for a given $\mathbf{r}$, the maximum c such that ex post efficient trade is possible, denoted by $c_n^{max}(\mathbf{r})$ (and defined to be $-\infty$ if no such c exists). Further, for each $c \leq \max_{\mathbf{r}} c_n^{max}(\mathbf{r})$, the boundary of the set of ownership structures such that ex post efficient trade is possible is defined by vectors $\mathbf{r}$ that satisfy (7) with equality. For example, if we consider a star or wheel network with $\mathbf{r} = (r, (1-r)/(n-1), \ldots, (1-r)/(n-1))$, then for each r and each $c \leq \max_{\mathbf{r}} c_n^{max}(\mathbf{r})$, we can calculate the maximum r, such that ex post efficient trade is possible, denoted by $\overline{r}_n(c)$. We illustrate this in Figure 2 for uniformly distributed types.

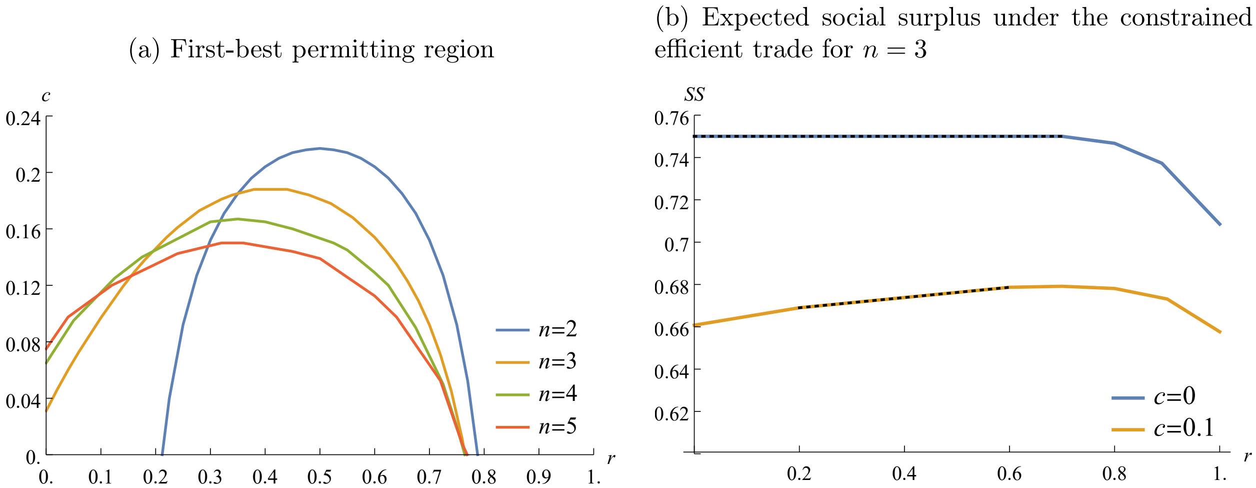

Figure 2: Maximum c that permits ex post efficient trade and expected social surplus for star networks. As illustrated, $\overline{r}_2(0) = 0.7887$, $\overline{r}_3(0) = 0.7654$, $\overline{r}_4(0) = 0.7647$, and $\overline{r}_5(0) = 0.7689$. Further, $c_2^{max} = 0.279$, $c_3^{max} = 0.187$, $c_4^{max} = 0.1675$, and $c_5^{max} = 0.150$. Assumes $\mathbf{r} = (r, \frac{1-r}{n-1}, \ldots, \frac{1-r}{n-1})$ and uniformly distributed types.

As illustrated in Figure 2, for a star network with $n = 2$ and $c = 0$, the first-best is possible for all $r \in [0.21, 0.79]$, which corresponds to the values obtained by Cramton et al. (1987). For $n = 2$, if $c > c_2^{max}$, then $\mathcal{R}(c)$ is empty (for uniformly distributed types, this occurs for $c > 0.279$), in which case the constrained efficient mechanism is used, implying that the ownership structure $\mathbf{r} = (1/2, 1/2)$ maximizes expected social surplus.

Proposition 5 raises the question of what ownership structure maximizes the budget surplus $\Pi^e_{\mathbf{r},C}$. We address this in the following proposition:

Proposition 6. *Given* c *that permits ex post e cient allocation, the ownership structure* $\tilde{\mathbf{r}} \in \mathcal{R}(c)$ *that maximizes* $\Pi^e_{\mathbf{r},C}$ *satisfies for* $i \in \{1, \ldots, n-1\}$*,*

$$\omega_i^e(r_i) = \omega_n^e(1 - \sum_{j=1}^{n-1} r_j) - \frac{dt^e_{(\mathbf{r}_{-n}, 1-r_1-\cdots-r_{n-1}),C}}{dr_i},$$

where $\frac{dt^e_{(\mathbf{r}_{-n}, 1-r_1-\cdots-r_{n-1}),C}}{dr_i} = \mathbb{E}_{\mathbf{v}}[\sum_{j=1}^n C_{ij}\hat{V}_{ij}(\mathbf{v}, C) - \sum_{j=1}^n C_{nj}\hat{V}_{nj}(\mathbf{v}, C)]$.

Proof. See Appendix A.

Proposition 6 says that the worst-off type of each firm $i \in \{1, \ldots, n-1\}$ must be equal to the worst-off type of firm n adjusted by the marginal impact of firm i's resource holdings on expected transportation costs. This implies that in the absence of transportation costs, $\tilde{\mathbf{r}}$ equalizes the worst-off types of all agents (Loertscher and Marx, 2022b, Lemma 2). For a star network, for example, $\tilde{\mathbf{r}}$ equalizes the worst-off types of the spokes, but that the worst-off type of the hub will be larger because expected transportation costs under the ex post efficient allocation decrease with the level of resources at the hub. Intuitively, to maximize budget surplus, we want the hub to be more of a seller than a buyer, so it should have a higher worst-off type than the agents in the spokes.[11]

While Figure 2 illustrates that $\overline{r}_n(0)$ need not be monotone in n, if one properly accounts for the expansion in the number of agents by calculating the resources accounted for by the first $x \in [0,1]$ share of agents, giving us a distribution of resources G_n defined by

$$G_n(x) \equiv \begin{cases} nx\overline{r}_n(0) & \text{if } x \leq 1/n, \\ 1-(1-x)n\frac{1-\overline{r}_n(0)}{n-1} & \text{if } x > 1/n, \end{cases}$$

then one finds that, at least for uniformly distributed types, G_n first-order stochastically dominates $G_{n'}$ if $n < n'$ (see Appendix B). Thus, with more agents, the boundary of $\mathcal{R}(0)$ shifts towards greater concentration at the hub.

3.3 Constrained efficient trade

An implication of Proposition 3 is that we must have either a nonextremal ownership structure or trade that is not ex post efficient, or both. And as illustrated above, for some transportation costs, ex post efficient trade is not an option for any ownership structure. Thus, we next characterize the constrained efficient mechanism.

The constrained efficient mechanism allocates resources to maximize the sum of the agents' expected surpluses subject to incentive compatibility, individual rationality, and no deficit, which requires that the expected budget surplus of the mechanism must be sufficient to cover the expected transportation costs.

To define the constrained efficient mechanism, it is useful to introduce the notion of weighted virtual types and their ironed counterparts. For $a \in [0,1]$, we denote by $\Psi_{i,a}(v;\hat{v})$ the weighted virtual type of agent i with type v and threshold type $\hat{v}$,

$$\Psi_{i,a}(v;\hat{v}) \equiv \begin{cases} \Psi^S_{i,a}(v) & \text{if } v \leq \hat{v}, \\ \Psi^B_{i,a}(v) & \text{if } v > \hat{v}, \end{cases}$$

where $\Psi^S_{i,a}(v) \equiv v + (1-a)\frac{F_i(v)}{f_i(v)}$ and $\Psi^B_{i,a}(v) \equiv v - (1-a)\frac{1-F_i(v)}{f_i(v)}$ are agent i's weighted

[11] For the star network with $n = 3$ and uniformly distributed types, the budget surplus maximizing r is $1/3$ for $c = 0$, and then it increases slightly as c increases, up to 0.4 for $c^{\max}$. As c increases, that maximization of budget surplus requires one to shift resources towards the hub—the reduction of information rents through symmetric ownership is gradually outweighed by the need to place resources at the hub in order to reduce transportation costs.

virtual cost and virtual value functions. (With identical distributions, we write $\Psi_a^S(v) \equiv v + (1-a)\frac{F(v)}{f(v)}$, $\Psi_a^B(v) \equiv v - (1-a)\frac{1-F(v)}{f(v)}$ and $\Psi_a(v;\hat{v})$ in lieu of $\Psi_{i,a}(v;\hat{v})$.) Although, as noted above, we assume that $\Psi_i^S(v)$ and $\Psi_i^B(v)$ are increasing, which implies that $\Psi_{i,a}^S(v)$ and $\Psi_{i,a}^B(v)$ are increasing for all $a \in [0,1]$, nevertheless, $\Psi_{i,a}(v;\hat{v})$ is not monotone, and so, as we shall see, will require ironing. We let $\overline{\Psi}_{i,a}(v;\hat{v})$ denote the ironed weighted virtual type of an agent with type v and threshold type $\hat{v}$, defined by

$$\overline{\Psi}_{i,a}(v;\hat{v}) \equiv \begin{cases} \Psi_{i,a}^S(v) & \text{if } \Psi_{i,a}^S(v) < z, \\ z & \text{if } \Psi_{i,a}^B(v) \leq z \leq \Psi_{i,a}^S(v), \\ \Psi_{i,a}^B(v) & \text{if } z < \Psi_{i,a}^B(v), \end{cases}$$

where the ironing parameter z satisfies

$$\int_0^{\hat{v}} \max\{0, \Psi_{i,a}^S(v) - z\} dF_i(v) = \int_{\hat{v}}^1 \max\{0, z - \Psi_{i,a}^B(v)\} dF_i(v). \tag{8}$$

The constrained efficient mechanism, as shown by Loertscher and Wasser (2019), is the solution to a saddle point problem that simultaneously chooses the allocation rule to maximize expected social surplus given agents' worst-off types, subject to constraints, and chooses the agents' worst-off types to minimize their expected payoffs given the allocation rule.

Focusing on the maximization problem for the moment, let ω_i denote agent i's worst-off type. Then letting ρ be the Lagrange multiplier on the no-deficit constraint and μ_i be the Lagrange multiplier on agent i's individual rationality constraint, and we have Lagrangian

$$\begin{aligned} \mathcal{L} \;\equiv\; & \mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^n (Q_i(\mathbf{v})v_i - Q_i(\mathbf{v})\Psi_i(v_i;\omega_i) + r_i\omega_i + u_i(\omega_i)) \right. \\ & \left. +\rho \left(\sum_{i=1}^n (Q_i(\mathbf{v})\Psi_i(v_i;\omega_i) - r_i\omega_i - u_i(\omega_i)) - T_{\mathbf{r},C}(\mathbf{Q}(\mathbf{v})) \right) \right] + \sum_{i=1}^n \mu_i u_i(\omega_i), \end{aligned}$$

where $T_{\mathbf{r},C}(\mathbf{Q}(\mathbf{v}))$ is the total transportation cost under allocation rule $\mathbf{Q}$ and type vector $\mathbf{v}$ when the ownership structure is $\mathbf{r}$ and transportation costs are given by matrix C. Rearranging this, we have

$$\mathcal{L} = \rho\mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^n Q_i(\mathbf{v})\Psi_{i,\frac{1}{\rho}}(v_i;\omega_i) - T_{\mathbf{r},C}(\mathbf{Q}(\mathbf{v})) \right] + (1-\rho)\sum_{i=1}^n r_i\omega_i + \sum_{i=1}^n (1-\rho+\mu_i)u_i(\omega_i).$$

Given $\boldsymbol{\omega}$ and ρ, we can then solve for $\mathbf{Q}$ pointwise, subject to the constraint that $\mathbf{Q}$ is nondecreasing (thus, requiring ironing). Specifically, given Lagrange multiplier ρ and worst-off types $\boldsymbol{\omega}$, the constrained efficient allocation rule for agent i is given by

$$Q^{ce}_{i,\mathbf{r},C}(\mathbf{v};\rho,\boldsymbol{\omega}) \equiv \sum_{j\in\mathcal{N}} \hat{V}^{ce}_{ji}(\mathbf{v},C;\rho,\boldsymbol{\omega})r_j,$$

where $\hat{V}^{ce}$ is defined analogously to $\hat{V}$, but with actual types replaced by ironed virtual types:

$$\hat{V}^{ce}_{ij}(\mathbf{v},C;\rho,\boldsymbol{\omega}) \equiv \begin{cases} 1 & \text{if } \overline{\Psi}_{j,1/\rho}(v_j;\omega_j) - c_{ij} \geq \max_\ell \overline{\Psi}_{\ell,1/\rho}(v_\ell;\omega_\ell) - c_{i\ell} \\ & \quad \text{and } \overline{\Psi}_{j,1/\rho}(v_j;\omega_j) - c_{ij} > \max_{\ell<j} \overline{\Psi}_{\ell,1/\rho}(v_\ell;\omega_\ell) - c_{i\ell}, \\ 0 & \text{otherwise.} \end{cases}$$

Using Lemma 2, the expected budget surplus under binding individual rationality is

$$\Pi^{ce}_{\mathbf{r},C}(\rho,\boldsymbol{\omega}) \equiv \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n} \Psi_i(v_i;\omega_i)Q^{ce}_{i,\mathbf{r},C}(\mathbf{v};\rho,\boldsymbol{\omega})\right] - \sum_{i=1}^{n} \omega_i r_i.$$

Given this, we can state the following result:

Proposition 7. *The constrained e cient allocation rule is the same as the ex post e cient allocation rule if* (7) *holds, and otherwise it is defined by* $\mathbf{Q}^{ce}_{\mathbf{r},C}(\mathbf{v};\rho^*,\boldsymbol{\omega}^*)$, *where* $\boldsymbol{\omega}^*$ *and* ρ^* *are such that for all* $i \in \mathcal{N}$, $\mathbb{E}_{\mathbf{v}_{-i}}[Q^{ce}_{i,\mathbf{r},C}(\omega^*_i,\mathbf{v}_{-i};\rho^*,\boldsymbol{\omega}^*)] = r_i$ *and* $\rho^* = \arg\min_\rho\{\rho \geq 1 \mid \Pi^{ce}_{\mathbf{r},C}(\rho,\boldsymbol{\omega}^*) \geq 0\}$.

The trading mechanism then has the allocation rule specified by Proposition 7 along with the payment rule given by Lemma 2, with $\boldsymbol{\omega}$ equal to $\boldsymbol{\omega}^*$. Figure 3 illustrates the contrast between the efficient and constrained efficient allocation rules for the case of two agents. In setups with identical distributions and no transportation costs, the optimal allocation rule coincides with the ex post efficient allocation when both agents have small and when both agents have large values if ironing occurs in the interior (see Loertscher and Wasser, 2019; Loertscher and Marx, 2022b). To see this, assume $v_1 > v_2$ and observe that under the optimal mechanism with $c = 0$ trade occurs if and only if $\Psi^S_a(v_1) > \Psi^S_a(v_2)$ when both types are small respectively $\Psi^B_a(v_1) > \Psi^B_a(v_2)$. With identical distributions this is equivalent to $v_1 > v_2$.

Interestingly, this feature does not extend to a settings with transportation costs, in which case it is easy to obtain, locally, more trade than under ex post efficiency. To see this, assume ironing occurs in the interior and consider v_1 and v_2 with $v_1 > v_2$, both of which are sufficiently small so that trade of r_2 occurs if and only if $\Psi^S_a(v_1) > \Psi^S_a(v_2) + c$, which is equivalent to

$$v_1 > v_2 + (1-a)\left[\frac{F(v_2)}{f(v_2)} - \frac{F(v_1)}{f(v_1)}\right] + c.$$

Under the constrained efficient mechanism, $a = 1/\rho^* < 1$, and so the right-hand side is smaller than $v_2 + c$—which is the condition for trade under ex post efficiency—if F/f is increasing. (And when both types are large, trade of r_2 occurs if and only if $v_1 > v_2 + (1 - a)\left[\frac{1-F(v_1)}{f(v_1)} - \frac{1-F(v_2)}{f(v_2)}\right] + c$, whose right-hand side is less than $v_2 + c$ if $(1-F)/f$ is decreasing.) These hazard rate properties are satisfied, for example, by the uniform distribution. This possibility of locally excessive trade is illustrated in Figure 3(c), which is plotted for the uniform distribution. At the boundaries, the orange contours of the constrained efficient allocation lie "inside" the blue contours for the ex post efficient allocation. Away from the boundaries, the constrained efficient allocation is shifted towards giving a greater quantity to the agent with greater initial resources (agent 1 in Figure 3), relative to the ex post efficient allocation.[12]

3.4 Trade-sacrifice mechanisms for networks

We now provide trade-sacrifice mechanisms that are prior-free. These mechanisms endow the agents with dominant strategies, respect their individual rationality constraints ex post, never run a deficit, and reallocate close ex post efficiently in a sense that will be made more precise below.

The constrained efficient mechanism used by the designer when ex post efficient trade is not possible makes use of the agents' type distributions and is thus not prior-free or, as Wilson (1987) put it, not detail-free. Building on Hagerty and Rogerson (1987), McAfee (1992), and Loertscher and Marx (2020b), we now provide ***trade-sacrifice mechanisms*** that are prior-free, endow the agents with dominant strategies, respect their individual rationality constraints ex post (and hence ad interim), never run a deficit, and either allocate ex post efficiently or "close to it." In our treatment, trade-sacrifice mechanisms are direct mechanisms that ask the agents to report their types. Our analysis applies for arbitrary ownership structures and for any network.

The trade-sacrifice mechanism is based on the ex post efficient trade. Despite seeming somewhat abstract, it is also useful to call agents buyers and sellers, under ex post efficiency,

[12]This possibility of locally excessive trade depends simultaneously on transportation costs and on the ironing ranges being interior. For example, if $c > 0$ is the fixed cost of producing a public good, in the optimal mechanism production occurs if and only if $\sum_i \Psi_a^B(v_i) > c$, which for any $a < 1$ is more restrictive than the condition for production under ex post efficiency. Likewise, if c is a transportation cost but ironing ranges are at the bounds, for example because $r_2 = 1$, trade occurs if and only if $\Psi_a^B(v_1) > \Psi_a^S(v_2) + c$, where for any any $a < 1$, the left-hand side is less than v_1 and the right-hand is larger than $v_2 + c$.

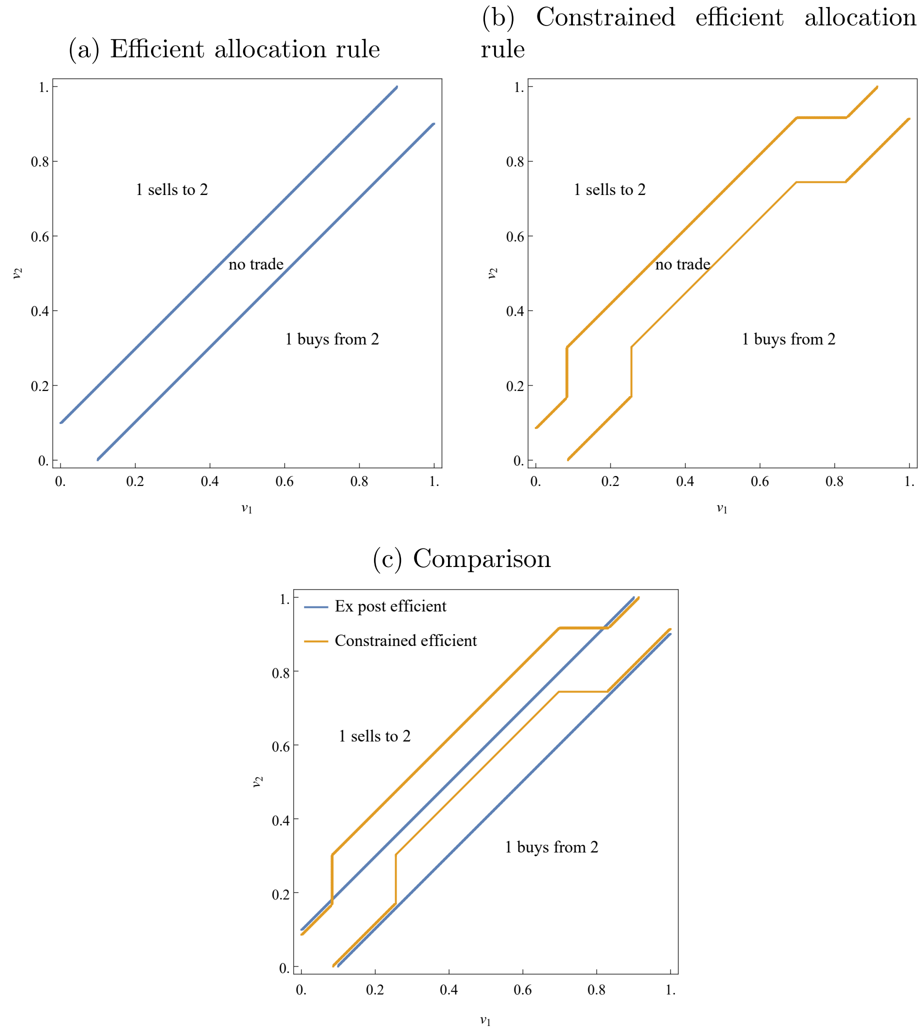

Figure 3: Assumes $n = 2$, $\mathbf{r} = (0.9, 0.1)$, $c = 0.1$, and uniformly distributed types. The constrained efficient functions are based on numerical calculations of $\rho^* = 1.18$ and ironing parameters $z_1^* = 0.8047$ and $z_2^* = 0.1953$.

irrespective of their resource holdings because this permits us to disentangle where the resources are from and who should obtain these under ex post efficiency. So in this terminology, agent i could be a seller even if $r_i = 0$, and agent j could be a buyer even if $r_j = 1$.

To understand the structure of ex post efficient trade on networks with costly transportation and to define the trade-sacrifice mechanism, it is useful to start with a star network. The star network has the property that, under ex post efficiency, there is a unique buyer whenever trade is ex post efficient, assuming away ties. (Even if there were ties, it would be without loss of generality to have a unique agent obtain all the resources that are traded.)[13]

[13]Thus, the star network preserves a property of the standard homogeneous good (or partnership) model without costly transportation in which all agents have constant marginal values up to the total supply available: all trade is directed to a single agent. This property does not hold, for example, for the wheel

Let $\mathbf{v}$ be the agents' types and let $k_{ij} = v_i + C_{ij}$ be agent i's effective per-unit cost of selling its resources r_i to agent j, which includes the transportation cost. Letting

$$\mathcal{S}^j(\mathbf{v}) \equiv \{i \mid k_{ij} < v_j\}$$

be the—possibly empty—set of agents with an effective cost of selling to agent j that is less than v_j, trade is ex post efficient if and only if $S^j(\mathbf{v}) \neq \emptyset$ for at least one $j \in \mathcal{N}$. Assuming trade is ex post efficient, let $b = \arg\max_{j\in\mathcal{N}} \sum_{i\in\mathcal{S}^j(\mathbf{v})}(v_j - k_{ij})$ be the agent who is the buyer and let $k_{hb} = \max_{i\in\mathcal{S}^b(\mathbf{v})} k_{ib}$ be the seller with the highest effective cost of serving b. It follows that any price $p^W \in [k_{hb}, v_b]$ will be market-clearing (or Walrasian) and support the efficient allocation: all agents in the set $\mathcal{S}^b(\mathbf{v})$ are willing to sell at such a price, understanding that they have to bear the transportation cost, and the buyer, agent b, is willing to buy.

For the star network, the trade-sacrifice mechanism that we propose asks the agents to report their types, and given the reports $\mathbf{v}$ determines b and k_{hb}. Agent h will be prevented from selling and all trade occurs at the price $p = k_{hb}$. Note that by construction, the mechanism always balances the budget. Relative to ex post efficiency, the trade it sacrifices is r_h, and so no trade is sacrificed if and only if $r_h = 0$. It is easy to see that the mechanism satisfies ex post individual rationality, and the demonstration that it endows the agents with dominant strategies follows along the usual lines—no agent can affect the price that it pays or is paid conditional on trading, and no agent that is prevented from trading (including agent h) can benefit from reporting a type such that it trades.

Consider now networks in general, in which case there may be multiple buyers under ex post efficiency. For every $j \in \mathcal{N}$, $\mathcal{S}^j(\mathbf{v})$ is defined as before. If $\mathcal{S}^j(\mathbf{v}) \cap \mathcal{S}^i(\mathbf{v}) \neq \emptyset$ for some $i \neq j$, eliminate agent $a \in \mathcal{S}^j(\mathbf{v}) \cap \mathcal{S}^i(\mathbf{v})$ from the set $\mathcal{S}^j(\mathbf{v})$ (and keep it in $\mathcal{S}^i(\mathbf{v})$) if and only if $v_i - v_j > k_{ai} - k_{aj} = C_{ai} - C_{aj}$. For all $i \in \mathcal{N}$, let $\hat{\mathcal{S}}^i(\mathbf{v})$ be the set obtained from $\mathcal{S}^i(\mathbf{v})$ after all such eliminations have been executed for all pairs of i and j, and assume that at least one of these sets is non-empty, which implies that trade is ex post efficient. The set of buyers under ex post efficiency given $\mathbf{v}$, denoted $\mathcal{B}(\mathbf{v})$, is then

$$\mathcal{B}(\mathbf{v}) = \arg\max_{\mathcal{J}\subset\mathcal{N}} \sum_{j\in\mathcal{J}} \left\{ \sum_{i\in\hat{\mathcal{S}}^j(\mathbf{v})} (v_j - k_{ij}) \right\}.$$

In analogy to the star network, for each $b \in \mathcal{B}(\mathbf{v})$, let $k_{h_b b} = \max_{i\in\hat{\mathcal{S}}^b(\mathbf{v})} k_{ib}$.[14]

network, where adjacent neighbors on the ring road sometimes trade with each other bilaterally under ex post efficiency. In contrast, in Loertscher and Marx (2020b), multiple agents may be buyers under ex post efficiency because the agents may have maximum demands that are less than the total supply.

[14]It may be natural to think that $\times_{b\in\mathcal{B}(\mathbf{v})}[k_{h_b b}, v_b]$ is the set of market-clearing (or Walrasian) prices.

For all $b \in \mathcal{B}(\mathbf{v})$, the trade-sacrifice mechanism prevents the agent with effective cost k_{hb} from trading and executes all trades to buyer b at the price $k_{h_b b}$, to be paid by the buyer, with the cost of transportation borne by the sellers. Relative to ex post efficiency, the trade that is sacrificed is r_{h_b}. As before, the mechanism balances the budget by construction, and for the same reasons as before, it respects the agents individual rationality constraints ex post and endows them with dominant strategies. We summarize these properties in the following proposition:

Proposition 8. *The trade-sacrifice mechanism balances the budget, respects the agents' individual rationality constraints ex post, and endows them with dominant strategies.*

The trade-sacrifice mechanism developed here, whose properties are stated in Proposition 8, is related to the one in Loertscher and Marx (2020b). The key differences in the setups are that there are no transportation costs there but the agents' maximum demands are allowed to be smaller than the total supply. As mentioned, this means that even with a homogeneous good multiple there may be multiple agents who buy under ex post efficiency. In the trade-sacrifice mechanism developed there, somewhat arbitrarily the agent with the lowest value who under ex post efficiency is allocated a positive amount is the agent whose type is the (reserve) price. Like here, this agent is prevented from trading. But because there may a long side of the market at the reserve price consisting of multiple agents, a Vickrey auction is used on the long side, which means that the mechanism may run a budget surplus.

While the trade-sacrifice mechanism is well-defined for any network and any ownership structure, its performance varies with the environment. For example, in a star network with agent 1 at the hub and $r_1 = 1$, trade only occurs under the trade-sacrifice mechanism if v_1 is not higher than the third-highest value draw and if the second-highest value is larger than $v_1 + c$. This raises the question of whether the mechanism could be improved upon, which we address next.

For a bilateral trade problem without transportation costs, that is, when $n = 2$ and $r_i = 1$ for some i and $c = 0$, Hagerty and Rogerson (1987) show that posting a price $p \in (0, 1)$ is the best a social-surplus maximizing designer can do, subject to dominant strategies, ex post individual rationality, and ex post budget balance, where posting a price p means that trade occurs if and only if the buyer reports a type above p and the seller reports a type below p; if trade occurs, the buyer pays p to the seller. With $n = 2$, there is never any trade under our trade-sacrifice mechanism for any $\mathbf{r}$, so posting a price $p \in (c, 1)$—with the understanding that the ex post seller bears the transportation cost and that trade of r_i units occurs (and

That is correct only if for all $b \in \mathcal{B}(\mathbf{v})$, $\hat{\mathcal{S}}^b(\mathbf{v}) = \mathcal{S}^b(\mathbf{v})$. Otherwise a seller who was part of $\mathcal{S}^b(\mathbf{v})$ and $\mathcal{S}^{b'}(\mathbf{v})$ may have any incentive to sell to b' given a price equal to $k_{h_b b}$ if it is supposed to sell to b under ex post efficiency.

if trade occurs, it takes place at price p per unit) if $v_i + c < p < v_j$—is evidently better than the trade-sacrifice mechanism above.

We show in Appendix E that, assuming that the underlying type distribution is known, augmenting the trade-sacrifice mechanism with a posted price greater than c does indeed increase expected social surplus relative to the baseline mechanism for all $c \in [0, 1)$. However, it is also the case that setting the posted price to be any fixed increment above c could result in a decrease in expected social surplus, depending on the distribution. Thus, it is not necessarily the case that one can improve upon the baseline mechanism in this way, at least not in a prior-free way. While one might consider estimating the distribution (Loertscher and Marx, 2020a, 2023), at that point the estimated distribution could be used in the context of the constrained efficient mechanism rather than the trade-sacrifice mechanism.

4 Optimal ownership structures for networks

The question of how optimally to place resources along a network before demand is realized, anticipating that subsequent transportation is costly, is fundamental to economics, ranging from the placement of medical and military equipment and personnel to the optimal location of production facilities within a firm or a country.[15] Two variants of the problem are of interest and reflect different degrees of power on behalf of the central authority in charge of the resource placement and its subsequent trade. Above we analyze the (social) ***planner's problem***, in which the central authority remains in control over the resources after they have been placed with some agent(s). In contrast, in what we call the (market) ***designer's problem***, resources placed at an agent's node bestow that agent with property rights or control over these resources. Consequently, the individual rationality constraints faced by the trade mechanism that ensues after demands are realized vary with the initial placement in the designer's problem but not in the planner's problem. Assuming that the trade mechanism must not run a deficit and has to respect the agents' individual rationality and incentive compatibility constraints, this means that the designer's problem is more constrained than the planner's.[16]

[15] Related to medical equipment, see, e.g., Simon Loertscher and Leslie M. Marx, "A National Ventilator Exchange Could Address Critical Shortages," *The Hill*, March 27, 2020. Related to military equipment, see, e.g., Daniel Michaels, "Ukraine War Spurs NATO to Improve Transport of Military Equipment," *Wall Street Journal*, January 5, 2023. Regarding production facilities, see Jon Hurdle, "These Sites No Longer Make Goods. Now They'll Get Them to You Faster," *New York Times*, December 18, 2018.

[16] Alternatively, and equivalently, one can describe the difference between the two problems as arising from the agents in the designer's problem having property rights over the resources at their nodes and private information about their demands, where the designer must respect the agents' incentive compatibility and individual rationality constraints. In contrast, in the planner's problem, the demands, once realized, are common knowledge.

In both the planner's and the designer's problem, the ownership structure has to be determined before the agents' types are realized, accounting for the possible type realizations and anticipating the cost of subsequent transportation. In the planner's problem, the placement of resources with an agent do not endow the agent with property rights, so trade in the planner's problem is a one-sided allocation problem in which the agents are buyers with outside options and worst-off types of 0. Because the planner can, for example, run a second-price auction, where each agent's value is adjusted for the required transportation cost, the trading phase always permits an incentive compatible and individually rational solution that does not run a deficit.[17] In contrast, in the designer's problem, the agents' outside option depends on the resources placed at their nodes. With extremal resource holdings, the trading phase is a two-sided allocation problem with one seller—the owner of the resources—and $n-1$ buyers. If the resources are dispersed, then the designer's allocation problem becomes what is sometimes called an "asset market" problem because the trading positions of the agents controlling the resources—buy, sell, or do not trade—depend, in general, on their own types and the types of all other agents.

We show that an extremal ownership structure followed by ex post efficient trade solves the planner's problem, raising question whether an extremal ownership structure followed by ex post efficient trade also solves the designer's problem. We next show that it does not, which implies that the first-best is possible in the designer's problem whenever an extremal ownership structure is uniquely optimal in the planner's problem.

4.1 Designer's problem

We are now in a position to characterize the optimal ownership structure in the designer's problem. As noted in Proposition 2, for a network with a unique completely connected agent, placing all resources with that agent is the uniquely optimal ownership structure in the planner's problem. However, an extremal ownership structure creates a tension with incentives, and this tension is at the center of the analysis that follows.

The results of this section establish that given a network with at least one completely connected agent: for $c > 0$ sufficiently small, the designer's problem is solved by different ownership structures from the planner's problem; for $c < 1$ sufficiently large, the designer's problem is solved by the same ownership structure as in the planner's problem, albeit with a different allocation rule; and for intermediate c, the designer's problem is solved by both different ownership structures and a different allocation rule than for the planner's problem.

It will be useful to define four sets of ownership structures. We suppress the specification

[17] By standard arguments, it can always be made to balance the budget; see e.g., Börgers and Norman (2009).

of the full transportation cost matrix C and simply parameterize the transportation cost by c. First, let $\mathcal{R}(c)$ be the set of ownership structures, possibly empty, satisfying (7). Second, let $\mathcal{R}^I(c)$ denote the set of ownership structures that minimize expected transportation costs subject to allowing ex post efficient trade, if such a ownership structures exists. Thus, for c such that $\mathcal{R}(c) \neq \emptyset$, we define $\mathcal{R}^I(c) \equiv \{\mathbf{r} \in \mathcal{R}(c) \mid \mathbf{r} \in \arg\min_{\mathbf{r}} t^e_{\mathbf{r},C}\}$. Third, we let $\mathcal{R}^P(c)$ denote the set of ownership structures that solve the planner's problem. And, fourth, we let $\mathcal{R}^D(c)$ denote the set of ownership structures that solve the designer's problem.

We consider the different levels of transportation cost in turn, beginning with a result for the case of zero transportation costs. In that case, any ownership structure that allows ex post efficient trade is optimal for the designer:

Proposition 9. *For* $c = 0$, *the designer's problem is solved by any* $\mathbf{r} \in \mathcal{R}(0)$, *i.e.,* $\mathcal{R}^D(0) = \mathcal{R}^I(0) = \mathcal{R}(0)$.

Further, using Proposition 2, for a network with at least one completely connected agent, for $c \in (0, 1)$, $\mathcal{R}^P(c)$ contains only extremal ownership structures, and using Proposition 3, $\mathcal{R}^I(0)$ does not contain any extremal ownership structure. Thus, by continuity, in such a network, for $c > 0$ sufficiently close to zero, the set of solutions to the designer's problem has an empty intersection with the set of solutions to the planner's problem:

Proposition 10. *For a network with at least one completely connected agent, there exists* $\hat{c} \in (0, 1)$ *such that for all* $c \in (0, \hat{c})$, $\mathcal{R}^D(c) \cap \mathcal{R}^P(c) = \emptyset$.

Turning to the case of sufficiently high transportation costs, we begin by noting that this case is simplified by each agent having the same worst-off type.

Lemma 3. *If* $c \geq 1/2$, *then* $1/2$ *is a worst-o type for every agent, i.e.,* $\omega_j(r) = 1/2$ *for all* $j \in \mathcal{N}$.

Proof. See Appendix A.

The maximized objective under the constrained efficient allocation rule can be written as:

$$\begin{aligned}\mathcal{L}^*(\mathbf{r}) \;\; &\equiv \;\; \rho^* \mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\Psi_{i,\frac{1}{\rho^*}}(v_i; \omega_i^*) - C_{ji} \right) \hat{V}_{ji}^{ce}(\mathbf{v}, C; \rho^*, \boldsymbol{\omega}^*) r_j \right] \\ &\quad + (1 - \rho^*) \sum_{j=1}^{n} \omega_j^* r_j + \sum_{j=1}^{n} (1 - \rho^* + \mu_j^*) u_j(\omega_j^*). \end{aligned} \tag{9}$$

For $c \geq 1/2$, we have $\omega_1^* = \cdots = \omega_n^* = 1/2$, so in that case $\sum_{j=1}^{n} \omega_j^* r_j = 1/2$, and the only direct effects of $\mathbf{r}$ occur in the expression in (9) in square brackets.

As the following lemma shows, we can rewrite the expectation of the term in square brackets in terms of the ironed rather than unironed weighted virtual types:

Lemma 4. *For $c \geq 1/2$ and $F_i = F$ for all $i \in \mathcal{N}$, the maximized objective $\mathcal{L}^*(\mathbf{r})$ can be written as:*

$$\mathcal{L}^*(\mathbf{r}) = \mathbb{E}_{\mathbf{v}}\Big[\sum_{j=1}^{n}\sum_{i=1}^{n}\Big(\overline{\Psi}_{i,\frac{1}{\rho^*}}(v_i;1/2) - c_{ji}\Big)\,\hat{V}^{ce}_{ji}(\mathbf{v},C;\rho^*,\mathbf{1/2})r_j\Big] + \frac{1-\rho^*}{2} + \sum_{j=1}^{n}(1-\rho^*+\mu_j^*)u_j(1/2).$$

Proof. See Appendix A.

Using Lemma 4, we see that the expression in square brackets in (9) is the same as the objective for the unconstrained problem (where the values appearing in the objective match the values appearing in the $\hat{V}$ matrix) with types drawn from the distribution of $\overline{\Psi}_{\frac{1}{\rho^*}}(v;\omega_i^*)$ rather than from F. Thus, by Proposition 2, we know that for a network in which the set of completely connected agents is nonempty, the objective is maximized by placing all the resources at one or more of the completely connected agents, giving us the following result:

Proposition 11. *Assuming that $F_i = F$ for all $i \in \mathcal{N}$, if the set of completely connected agents $\mathcal{I}$ is nonempty, then there exists $\hat{c} \in [1/2, 1)$ such that for all $c \geq \hat{c}$ and $\mathbf{r} \in \mathcal{R}^D(c)$, we have $r_i = 0$ for all $i \in \mathcal{N}\backslash\mathcal{I}$.*

As shown above in the discussion of Figure 2, when $n = 2$ and c is sufficiently large that ex post efficient trade is not possible for the designer, then $\mathbf{r} = (1/2, 1/2)$ is the designer's optimal ownership structure. Extending this to general n, we have the following result:

Proposition 12. *Given a complete network with $F_i = F$ for all $i \in \mathcal{N}$, $\mathbf{r} = (1/n, \ldots, 1/n)$ is optimal for the designer for any $c > \max_{\mathbf{r}} c_n^{max}(\mathbf{r})$.*

Proof. Let $r_n = 1 - r_1 - \cdots - r_{n-1}$ and differentiate L^* given in (9) with respect to r_j for $j \in \{1, \ldots, n-1\}$. Focusing on direct effects, we obtain a first-order condition that is zero when $\mathbf{r} = (1/n, \ldots, 1/n)$. ■

Proposition 12 implies that the constrained efficient mechanism will be used for symmetric ownership, which is both interesting and new and useful because it makes the analysis of the problem (more) tractable.

In cases in which ex post efficient trade is possible, we have $\rho^* = 1$, and so

$$\mathcal{L}^*(r) = \mathbb{E}_{\mathbf{v}}\Big[\sum_{j=1}^{n}\sum_{i=1}^{n}(v_i - C_{ji})\,\hat{V}_{ji}(\mathbf{v},C)r_j\Big] + \sum_{j=1}^{n}\mu_j^* u_j(\omega_j^*),$$

where again the term in square brackets is the objective for the unconstrained problem and so solved with extremal ownership structures, giving us the following result:

Proposition 13. *For a star or wheel network with solution to the designer's problem of* $\mathbf{r}^* = (r, \frac{1-r}{n-1}, \ldots, \frac{1-r}{n-1})$ *and* $F_i = F$ *for all* $i \in \mathcal{N}$*, if* $\mathbf{r}^* \in \mathcal{R}(c)$*, then* $r_1^* = \max\{r_1 \mid (r_1, \mathbf{r}_{-1}) \in \mathcal{R}(c)\}$*.*

This says that for a star or wheel network, when the solution to the designer's problem involves ex post efficient trade, then the designer's optimal ownership structure is on the boundary of the region permitting ex post efficient trade.

Combining these results, we see that for a star or wheel network, for a range of intermediate values for c, the solution to the designer's problem has a ownership structure $\mathbf{r}^*$ that does not permit ex post efficient trade, and is not extremal, and so the solution is intermediate between the solution to the designer's problem with $c = 0$ and with $c \geq 1/2$. We illustrate this in Figure 4.

(a) Graph of $\mathcal{R}(c)$ and $r^D(c)$ for a $n = 3$ star network

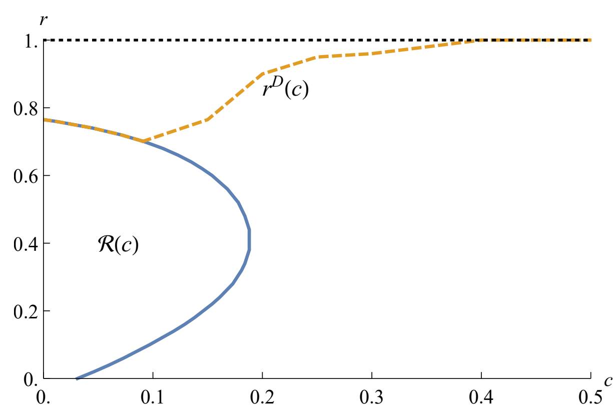

Figure 4: Maximum c that permits the first-best and $r^D(c)$ for a star network. Assumes $\mathbf{r} = (r, \frac{1-r}{n-1}, \ldots, \frac{1-r}{n-1})$ and uniformly distributed types.

Wheel networks

For wheel networks, by Proposition 2, the solution to planner's problem is the same as for a star network and involves placing all resources with the hub. But the solution to the designer's problem will differ between the two network structures because in a wheel network, peripheral agents can trade with their immediate neighbors if they have positive endowments. As a result, for wheel networks and small c, ex post efficient trade is possible with a more extreme (closer to the planner's optimum) ownership structure, i.e., increased resources at the hub. We illustrate the contrast in Figure 5, which shows that for c close to zero and $n = 5$, the region permitting ex post efficient trade for the wheel network includes larger values of r than does the corresponding region the star network.

(a) Maximum c that permits the first-best

(b) Maximum c that permits the first-best

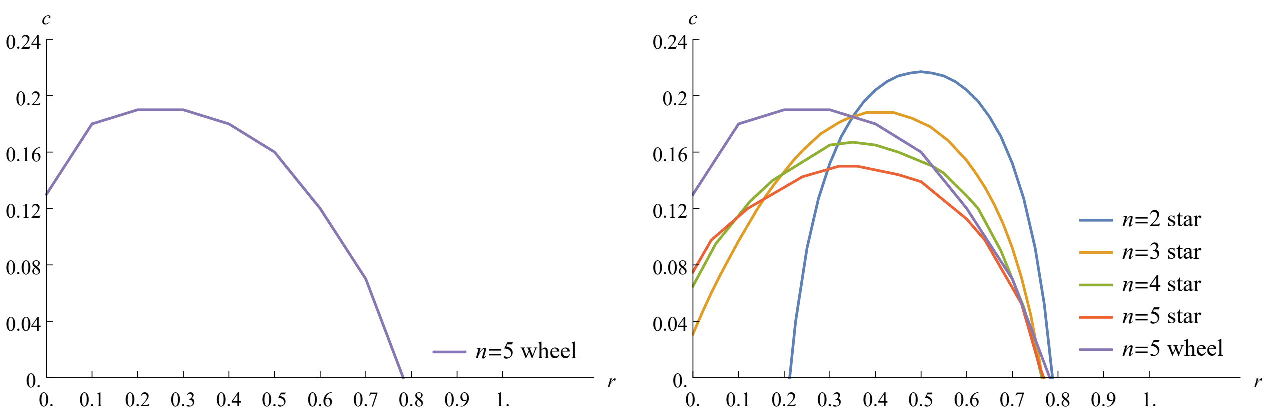

Figure 5: Maximum c that permits ex post efficient trade for a wheel network and also contrasted with star networks. For the $n = 5$ wheel, $c^w_{max} = 0.19$ and $\overline{r}^w_5(0) = 0.782$. Assumes $\mathbf{r} = (r, \frac{1-r}{n-1}, \ldots, \frac{1-r}{n-1})$ and uniformly distributed types.

Results for a wheel network emphasize that having more trades does not mean that a market achieves greater efficiency. For example, for the wheel network with $n = 5$ and $c = 0.05$, the planner's optimal ownership structure has all the resources at the hub. In this case the probability of there being no trade is approximately 25%, and the expected number of trades is approximately 0.75. In contrast, for a designer with $r = 0.7280$, which is on the boundary of the region permitting ex post efficient trade, the probability of no trade is 0.01% and the expected number of trades is approximately 0.77. While the planner delivers the more efficient outcome, there are more trades under the designer.

Table 1: Wheel network outcomes for the planner with $\mathbf{r} = (1, 0, 0, 0, 0)$ and the designer with $\mathbf{r} = (r, (1 - r)/(n-1), \ldots, (1-r)/(n-1)$ with $r = 0.7280$, followed by ex post efficient trade. Assumes $n = 5$, $c = 0.05$, and uniformly distributed types. Based on a simulation using 10,000 draws of $\mathbf{v} \in [0, 1]^5$.

	no trade	only 1 agent is a buyer	2 agents are buyers	expected trades
planner	25.12%	74.88%	0	0.7488
designer	0.01%	87.95%	12.04%	0.7662

In the planner's problem, the optimal ownership structure has resources at the hub, so the possibilities are that we have no trade at all or one trading cluster whereby the hub sells to the spoke with the highest value. In contrast, for the designer with $r \in (0, 1)$, we can have zero, one, or two trading clusters. These correspond to whether there are zero, one, or two agents that are allocated more than their initial holdings, which we refer to here as "buyers." In an $n = 5$ wheel, it is not possible to have more than two buyers. As illustrated in Table 1, in the designer's problem, of the 88% of cases in which only one agent was a buyer, those can be divided into 20% of cases in which the trading cluster involved the hub and 68% in which trading was along the ring road only. The 12% of cases in which there were two trading clusters, can be divided into 4% that had one cluster involving the hub and one along the ring road and 8% where there were two trading clusters, neither of which involved the hub, implying that there were two separate trading clusters both along the ring road. We illustrate a wheel network with two trading clusters in Figure 1.

4.2 Optimal ownership structure anticipating trade via trade-sacrifice mechanism

Under the trade-sacrifice mechanism, social surplus is

$$\sum_{i \in \mathcal{S}^b(\mathbf{v}) \setminus \{h\}} (v_b - C_{ib}) r_i + \sum_{i \in \mathcal{N} \setminus \mathcal{S}^b(\mathbf{v})} v_i r_i + v_h r_h.$$

Consider the optimal ownership structure, in the sense of maximizing expected social surplus, anticipating trade using the trade-sacrifice mechanism. We focus on the star network with ownership structure $\mathbf{r} = (r, \frac{1-r}{n-1}, \ldots, \frac{1-r}{n-1})$. Because expected social surplus is linear in r, the optimal r will be either zero or one, or expected social surplus will be constant in r. Because the trade-sacrifice mechanism sacrifices one trade, one might expect that it would deliver more efficient outcomes when resources are dispersed among the spokes of the network ($r = 0$), thereby diminishing the importance of the excluded agent. Indeed, this is the case for sufficiently large c:

Proposition 14. *Assuming that trade occurs via the baseline trade-sacrifice mechanism, for a star network,* $r = 0$ *maximizes expected social surplus if* $c \geq 1/2$.

Proof. Under the trade-sacrifice mechanism, there is only trade if there is a buyer b with $r_b < 1$ and at least two sellers, i.e., $|S^b(\mathbf{v})| \geq 2$. For $c \in [1/2, 1)$, a spoke agent never satisfies the condition to be a seller to another spoke agent, so the only possibility to satisfy the conditions for trade is if the hub is the buyer, two or more spoke agents are sellers, and $r < 1$ (so that the buyer has demand for additional units). If $r = 1$, there is no trade, and so expected social surplus is equal to $\mathbb{E}_v[v]$. If $r < 1$, then there is social surplus enhancing trade for some type realizations, resulting in expected social surplus that is greater than $\mathbb{E}_v[v]$. It then follows from the linearity of expected social surplus in r that $r = 0$ maximizes expected social surplus for $c \in [1/2, 1)$. ■

While Proposition 14 provides conditions under which the optimal ownership structure for the trade-sacrifice mechanism has $r = 0$, that is not always the case. Recall that the planner optimally has $r = 1$ because that minimizes the expected transportation costs associated with ex post efficient trade. For the trade-sacrifice mechanism, as for the ex post efficient trade, a ownership structure with $r = 1$ has the advantage that the market never incurs the transportation cost from one spoke to another of $2c$. However, for the trade-sacrifice mechanism, a ownership structure with $r = 1$ has the disadvantage that trade only occurs when $v_1 + c < v_j + 2c < v_i$ for some $i, j \in \{2, \ldots, n\}$ with $i \neq j$ because that is required for the hub to trade as a seller. When c is small, this condition is less restrictive. For example, as we show in Appendix D, for $n = 3$ and uniformly distributed values, $r = 1$ uniquely maximizes expected social surplus for $c \in (0, 0.3218)$, and $r = 0$ uniquely maximizes expected social surplus for $c \in (0.3218, 1)$.

Proposition 14 shows that with the trade-sacrifice mechanism, for a star network with $c \geq 1/2$, the optimal placement is $r = 0$. This occurs because when $r = 1$, trade can never occur in our "seller-side" trade-sacrifice mechanism—even if a spoke is the buyer, the cost of $c \geq 1/2$ prevents there from being more than one seller. For a star network, it was natural to define a seller-side mechanism because a star has a uniquely defined buyer. However, for $c \geq 1/2$ and a star network with $r = 1$, we can alternatively identify the hub as the unique seller and define a "buyer-side" trade-sacrifice mechanism to have trade between between the hub and spoke b with $v_b = \max\{v_2, \ldots, v_n\} > v_1 + c$ whenever there exists another spoke h with $v_b > v_h > v_1 + c$. In that case, trade can occur at price $v_h - c$, with the buyer covering the transportation cost. In this way, the trade-sacrifice mechanism can be customized based on the observed network and ownership structure, which can be done without knowledge of the type distribution. However, to determine the optimal ownership structure, the expected benefits of $r = 0$ combined with a seller-side trade-sacrifice mechanism versus $r = 1$ combined with a buyer-side trade-sacrifice mechanism depend on underlying distributions.[18] Thus, in a prior-free setting, we cannot identify a clear optimum.

We conclude this section with a discussion of an open question regarding trade-sacrifice mechanisms for general networks. For general networks, under ex post efficiency there may be any number of clusters of trading agents, each constituting a seemingly independent market, endogenously determined by agents' values and transportation costs. It may then seem natural to extend the trade-sacrifice mechanism of Loertscher and Marx (2020b) by preventing the marginal seller in each cluster from selling and using its effective cost k_h as

[18]For example, with $n = 3$ and $r = 0$, there is trade in the seller-side trade-sacrifice mechanism only if $v_1 > \max\{v_2, v_3\} + c$. In contrast, if $r = 1$, there is trade in the buyer-side trade-sacrifice mechanism only if $\min\{v_2, v_3\} - c > v_1$. For general distributions, either could be greater.

the price in that cluster. But that fails to be incentive compatible because the marginal seller in one cluster may then want to sell into another cluster.[19] A possible solution would be to define prices as the maximum across "adjacent" clusters, but it is not clear where, in general, to draw the lines.

5 Extensions

In this section, we provide extensions to allow, in turn, for costly entry, the presence of an "empty hub," and heterogeneous distributions.

5.1 Entry

As is, the "market economy"—i.e., the problem the designer faces—always performs worse than the centrally planned economy. We now extend the model with a star network to allow for positive costs of entering at a node. This allows us to consider the question whether competition better stimulates entry, which one might think of as innovation. Hayek (2002, p. 16) remarks on an advantage of a planned economy over a market economy in that in the market economy "as much is produced as we can manufacture by any method that is known to us," but that market economy output "is of course not as much as we could produce if in fact all the knowledge that anyone possessed or could acquire were available at a central point."[20] While Hayek recognizes that the market economy falls short of the theoretical frontier, he also comments on its value for spurring innovation: "competition is important primarily as a discovery procedure whereby entrepreneurs constantly search for unexploited opportunities that can also be taken advantage of by others" (Hayek, 2002, p. 18).

Take the case of an $n = 3$ star network. There is a hub, but spokes can either be occupied or not. The planner's optimal ownership structure has all resources at the hub. In contrast, the designer's optimal ownership structure has resources at the hub equal to $r_n^D(c)$, i.e., if

[19] As an example, consider the following problem with $n = 6$ and $v_1 = 10$ and $v_2 = 9.8$, where agents 1 and 2 are buyers. For $i \in \{3, \ldots, 6\}$, let $v_i = i$ and assume that $C_{i1} = 1$ if i is odd and $C_{i1} = 1.5$ if i is even, and assume that $C_{i2} = 1$ if i is even and $C_{i2} = 1.5$ if i is odd (which can thought of as a variant of a Hotelling model). Under ex post efficiency, the odd-numbered agents sell to buyer 1, and the even-numbered ones sell to buyer 2. In the trade-sacrifice mechanism that blindly applies the mechanism sketched above, we have $p_O = 6(= 5 + 1)$ in the odd cluster and $p_E = 7(= 6 + 1)$ in the even cluster. But given these prices, agent 5 prefers to sell in the even cluster because $p_E = 7 > 6.5 = 5 + 1.5 = k_{5,2}$, where $k_{5,2}$ is agent 5's effective cost for selling to buyer 2.

[20] "A mind endowed with full information could of course choose every point on the n-dimensional surface that appeared desirable to him and then distribute as he saw fit the product of the combination he chose. But the only point on (or at least somewhere near) that surface we can reach using a procedure known to us is the one we reach when we leave its determination up to the market" (Hayek, 2002, p. 16).

no spokes are occupied, then $r_1^D(c) = 1$ is placed at the hub, if one spoke is occupied, then $r_2^D(c)$ is placed at the hub, and if both spokes are occupied, then $r_3^D(c)$ is placed at the hub.

We contrast entry under the planner versus entry under the designer, assuming that the buyer pays the transportation costs for items that it consumes above and beyond its initial holdings. In this setup, agent i's expected payoff is

$$\pi_i = \mathbb{E}_{\mathbf{v}}\big[\sum_{j=1}^{n} (v_i - C_{ji})\,\hat{V}_{ji}(\mathbf{v}, C) r_j\big].$$

Consider the case of an $n = 4$ star with $c = 0.05$, for which the designer achieves the ex post efficient trade (for additional details, see Appendix C).

Table 2: Hub and spoke expected payoffs as a function of the number of occupied spokes

	planner		designer		
occupied spokes	hub payoff	spoke payoff	$r_n^D(c)$	hub payoff	spoke payoff
0	0.5		1	0.5	
1*	0.3571	0.2858	0.7708	0.3408	0.3021
2	0.2821	0.2179	0.7365	0.2687	0.2228
3	0.2363	0.1758	0.7250	0.2252	0.1775

Assumes an $n = 4$ star network, $c = 0.05$, and uniformly distributed types.

*The network with 1 spoke is completely connected and so both the planner and the designer could optimally choose $r_1 = r_2 = 1/2$, in which case ex post efficient trade is possible even for the designer..

Thus, depending upon entry costs, we get different numbers of spokes, as shown in the figure below. For c small, centralization of initial holdings by the planner can induce further centralization in the sense that fewer spokes enter. In a sense, for c small, the property rights endowed to agents in the designer's problem have the potential to induce excessive entry. For c large, we have $r^D(c) = 1$, implying that the designer's optimal ownership structure also has all resources at the hub, but then the spokes have lower payoffs under the designer because the market is less efficient, and so entry is weakly greater under the planner—in that case, the efficiency of the planner can induce socially wasteful entry.

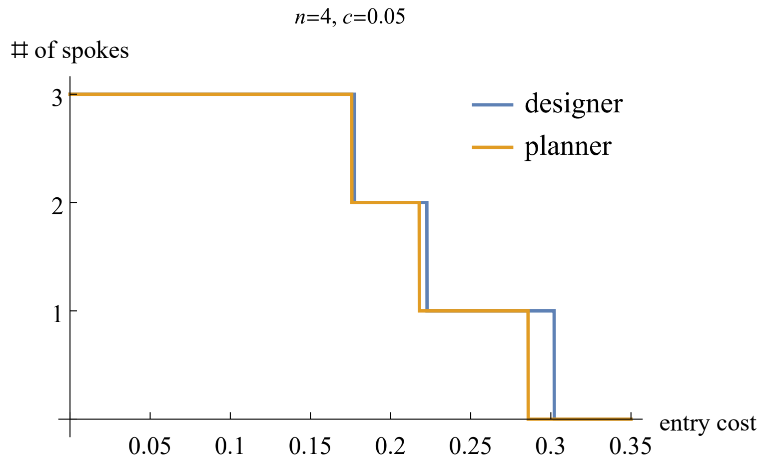

Figure 6: Entry of spokes as a function of the entry cost, for both the designer's problem and the planner's problem. Assumes an $n = 4$ star network, $c = 0.05$, and uniformly distributed types.

5.2 Empty hub

A question that arises in the design of distribution networks is whether there might be incentives to locate a hub in a place where there is no demand.[21] So consider the case of a star network with an "empty hub" in the sense that it is common knowledge that the hub has value zero for the resource. In this case, for $c \geq 1/2$, the planner's problem and designer's problem are the same because goods are only ever transported from the hub to a spoke, and spokes can be induced to reveal their values truthfully simply by charging them the maximum of the cost of transportation from the hub and the second-highest reported type, implying that ex post efficient trade is possible for the designer. Further, if it is optimal for the planner to place all resources at an empty hub, then it is also optimal for the designer to do so because then, once again, ex post efficient trade is possible for the designer. Given this, in what follows we focus on the planner's problem.

Consider ownership structures of the form $\mathbf{r} = (r, (1-r)/(n-1), \ldots, (1-r)/(n-1))$, where r is the resource holding at the empty hub. Because the expected social surplus under the ex post efficient allocation rule is linear in r, the planner's optimal ownership structure has either $r = 0$ or $r = 1$. Comparing the expected social surplus from these two options, we have the following lemma:

Lemma 5. *For a star network with an empty hub, the planner's optimal ownership structure has $r = 0$ if*

$$\mathbb{E}_v[v] \geq \mathbb{E}_{\mathbf{v}}\left[\max\{0, v_{(1)} - c\}\right] - \mathbb{E}_{\mathbf{v}}\left[\max\{0, v_{(1)} - v_n - 2c\}\right]. \tag{10}$$

[21]For an example, see Josh Dzieza, "The Everything Town in the Middle of Nowhere: How the Tiny Town of Roundup, Montana, Became a Hub in Amazon's Supply Chain," *The Verge*, November 14, 2019.

and $r = 1$ otherwise.

Proof. See Appendix A.

Using Lemma 5, we have the following result:

Proposition 15. *For a star network with an empty hub and $c \in (0, 1)$, the planner's optimal ownership structure has $r = 0$ if $n = 2$, but has $r = 1$ if c is su ciently close to zero and n is su ciently large. Further, if $\mathbb{E}_v[v] \geq 1/2$, then for all $c \in [1/2, 1]$ and $n \geq 2$, the planner's optimal ownership structure has $r = 0$; whereas, if $\mathbb{E}_v[v] < 1/2$, then for $c \in [1/2, 1 - \mathbb{E}_v[v])$ and n su ciently large, the planner's optimal ownership structure has $r = 1$.*

Proof. The proof follows from Lemma 5. For details, see Appendix A.

Proposition 15 implies that for a planner (or designer) facing a star network with an empty hub, for some distributions and costs, no resources are placed at the hub, regardless of the number of spokes, but for other distributions and costs, the optimal ownership structure depends on the number of spokes, with low numbers of spokes resulting in no resources being placed at the hub, and high numbers of spokes resulting in all resources being placed at the hub.

Applying Proposition 15 to the uniform distribution, we have the immediate result that the planner's and designer's optimal ownership structure for $c \geq 1/2$ has $r = 0$, regardless of the number of spokes. For $c \in (0, 1/2)$, and uniformly distributed values, we can write inequality (10) as:[22]

$$\frac{1}{2} \geq \frac{1}{2} + \frac{n-3}{n-1}c - 2c^2 + \left(\frac{2^n}{n(n-1)} + \frac{1}{n}\right) c^n.$$

which is satisfied for all $n \in \{2, 3, \dots\}$ for $c = 1/2$,[23] and is satisfied for all $c \in [0, 1/2]$ for $n \in \{2, 3\}$, but for $c \in (0, 1/2)$, there exists n sufficiently large that it is not satisfied,[24] giving us the following corollary:

Corollary 3. *For a star network with an empty hub and uniformly distributed types, the ownership structure with $r = 1$ is optimal for the planner if and only if $c \in [0, 1/2)$ and n is su ciently large.*

[22]For the uniform distribution and $c \in (0, 1/2)$, we have $\mathbb{E}_\mathbf{v}\left[\max\{0, v_{(1)} - v_n - 2c\}\right] = \frac{n-2}{2n} - 2\frac{n-2}{n-1}c + 2c^2 - \frac{2^n}{n(n-1)}c^n$ and $\mathbb{E}_\mathbf{v}\left[\max\{0, v_{(1)} - c)\}\right] = \int_c^1 (x - c)(n-1)x^{n-2}dx = \frac{n-1}{n} - c + \frac{1}{n}c^n$.

[23]For $c = 1/2$, the right side is increasing in n and has limit equal to $1/2$ as n goes to infinity.

[24]To see this, note that the limit as n goes to infinity of the right side is $\frac{1}{2} + c - 2c^2 + \lim_{n\to\infty} \frac{2^n c^n}{n(n-1)} + \lim_{n\to\infty} \frac{c^n}{n} = \frac{1}{2} + c - 2c^2$, which is greater than $1/2$ for $c \in (0, 1/2)$.

With an empty hub, the planner's optimal ownership structure has $r = 1$ for the uniform distribution and, for example, $c = 0.1$ if and only if $n \geq 4$. For an example with $c \geq 1/2$, consider the distribution $F(x) = 1 - (1-x)^2$, in which case $\mathbb{E}_v[v] = 1/3$, and $c = 1/2$. Then the planner optimally sets $r = 1$ if and only if $n \geq 29$.

5.3 Heterogeneous distributions

This section provides an extension to allow for heterogeneous type distributions.

For our extension to heterogeneous type distributions, we focus on the case of $n = 2$ agents. Let $r_1 = r$, so that $r_2 = 1 - r$. The ex post efficient allocation is $Q_1(\mathbf{v}) = 1$ if $v_1 \geq v_2 + c$, $Q_1(\mathbf{v}) = r$ if $v_2 - c \leq v_1 \leq v_2 + c$, and $Q_1(\mathbf{v}) = 0$ otherwise, while $Q_2(\mathbf{v}) = 1 - Q_1(\mathbf{v})$. Social surplus under the ex post efficient allocation rule is $v_1 - (1-r)c$ if $v_1 \geq v_2 + c$, $rv_1 + (1-r)v_2$ if $v_2 - c \leq v_1 \leq v_2 + c$, and $v_2 - rc$ otherwise. Ex ante expected welfare given r and anticipating ex post efficient trade, denoted $SS^e(r)$, is

$$\begin{aligned} SS^e(r) \;\equiv\; & (\mathbb{E}_{\mathbf{v}}[v_1 \mid v_1 \geq v_2 + c] - (1-r)c) \Pr(v_1 \geq v_2 + c) \\ & + (\mathbb{E}_{\mathbf{v}}[v_2 \mid v_2 \geq v_1 + c] - rc) \Pr(v_2 \geq v_1 + c) \\ & + (r\mathbb{E}_{\mathbf{v}}[v_1 \mid v_2 - c \leq v_1 \leq v_2 + c] + (1-r)\mathbb{E}_{\mathbf{v}}[v_2 \mid v_1 - c \leq v_2 \leq v_1 + c]) \\ & \cdot \Pr(v_2 - c \leq v_1 \leq v_2 + c), \end{aligned}$$

the derivative of which is

$$\begin{aligned} SS^{e\prime}(r) = & \; c\,(\Pr(v_1 \geq v_2 + c) - \Pr(v_2 \geq v_1 + c)) \\ & (\mathbb{E}_{\mathbf{v}}[v_1 \mid v_2 - c \leq v_1 \leq v_2 + c] - \mathbb{E}_{\mathbf{v}}[v_2 \mid v_1 - c \leq v_2 \leq v_1 + c]) \\ & \cdot \Pr(v_2 - c \leq v_1 \leq v_2 + c). \end{aligned}$$

If the expressions in the first and the second line have the same sign, then the following (rather surprising) result holds:

Lemma 6. *If* $\Pr(v_1 \geq v_2 + c) - \Pr(v_2 \geq v_1 + c)$ *has the same nonzero sign as* $\mathbb{E}_{\mathbf{v}}[v_1 \mid v_2 - c \leq v_1 \leq v_2 + c] - \mathbb{E}_{\mathbf{v}}[v_2 \mid v_1 - c \leq v_2 \leq v_1 + c]$*, then for any* $c \in [0,1]$*, the solution to the planner's problem involves an extremal ownership structure, i.e., either* $\mathcal{R}^P(c) = \{(0,1)\}$ *or* $\mathcal{R}^P(c) = \{(1,0)\}$*.*

Note that the expressions are both equal to zero if $F_1 = F_2$, in which case any $r \in [0,1]$ is optimal, including the boundary conditions. But the expressions will have the same non-zero sign much more generally, including whenever if $F_1(v) = v$ and $F_2(v) = 1 - (1-v)^a$ for $a > 0$, in which case only extremal ownership structures solve the planner's problem.

To provide more general conditions, let v_1 have the density f and v_2 have the density g and suppose these two random variable satisfy the ***monotone likelihood ratio property (MLRP)***, that is, for any $x > y$ with $x, y \in [0,1]$, we have

$$\frac{f(x)}{g(x)} > \frac{f(y)}{g(y)}. \tag{11}$$

Note that MLRP implies, for all $x \in [0,1]$, both

$$\frac{G(x)}{g(x)} > \frac{F(x)}{f(x)} \quad \text{and} \quad \frac{1-G(x)}{g(x)} < \frac{1-F(x)}{f(x)},$$

i.e., hazard and reverse hazard rate dominance, each of which implies $F(x) \leq G(x)$ for all $x \in [0,1]$, i.e., FOSD.[25]

Using the MLRP condition and Lemma 6, we can then prove the following result:

Proposition 16. *For $n = 2$ and $F_1 = F_2 = F$, under MLRP and for $c > 0$ su ciently small, the solution to the planner's problem involves an extremal ownership structure.*

Proof. In Appendix A, we first show that MLRP implies that $\Pr(v_1 \geq v_2 + c) - \Pr(v_2 \geq v_1 + c) > 0$, and then we show that MLRP plus $c > 0$ sufficiently small implies that $\mathbb{E}_{\mathbf{v}}[v_1 \mid v_2 - c \leq v_1 \leq v_2 + c] - \mathbb{E}_{\mathbf{v}}[v_2 \mid v_1 - c \leq v_2 \leq v_1 + c] > 0$.

As the proof reveals, given MLRP, requiring c sufficiently small is only a simple sufficient condition. We are not aware of an example satisfying MLRP in which the result is reversed for some c.

5.4 Augmenting the model to include bargaining weights

We illustrate Propositions 3 and 4 in Figure 7, which considers the augmented model to allow bargaining weights. Specifically, we use the model of incomplete information bargaining of Loertscher and Marx (2022a) in which each agent i is endowed with bargaining weight $w_i \in [0,1]$, where $w_i > 0$ for at least one agent.[26] Using these bargaining weights, we

[25]To see that MLRP implies reverse hazard rate dominance, rearrange (11) to get

$$f(x)g(y) > f(y)g(x). \tag{12}$$

Then integrate to obtain $\int_0^x f(x)g(y)dy = f(x)G(x) > g(x)F(x) = \int_0^x f(y)g(x)dy$, which means that $\frac{G(x)}{g(x)} > \frac{F(x)}{f(x)}$. An analogous argument can be invoked to establish that $\frac{1-G(x)}{g(x)} < \frac{1-F(x)}{f(x)}$.

[26]In the case of equal bargaining weights, we assume that any budget surplus is distributed to the agents equally.

adjust the market mechanism to be the mechanism that maximizes ***weighted*** expected social surplus,

$$\sum_{i=1}^{n} w_i \left(Q_i(\mathbf{v})v_i - M_i(\mathbf{v})\right),$$

subject to incentive compatibility, individual rationality, and no deficit.[27]

Figure 7(a) illustrates Proposition 3, showing that when $r_1 = 1$, then ex post efficient trade is not possible for any bargaining weights. However, as shown in the figure, for $\mathbf{r} = (1/2, 1/2)$ and $c = 0$, the ex post efficient trade is possible for equal bargaining weights. Figure 7(b) illustrates Proposition 4, showing that with $c = 1/2$, regardless of the ownership structure and regardless of the bargaining weights, ex post efficient trade is not possible.

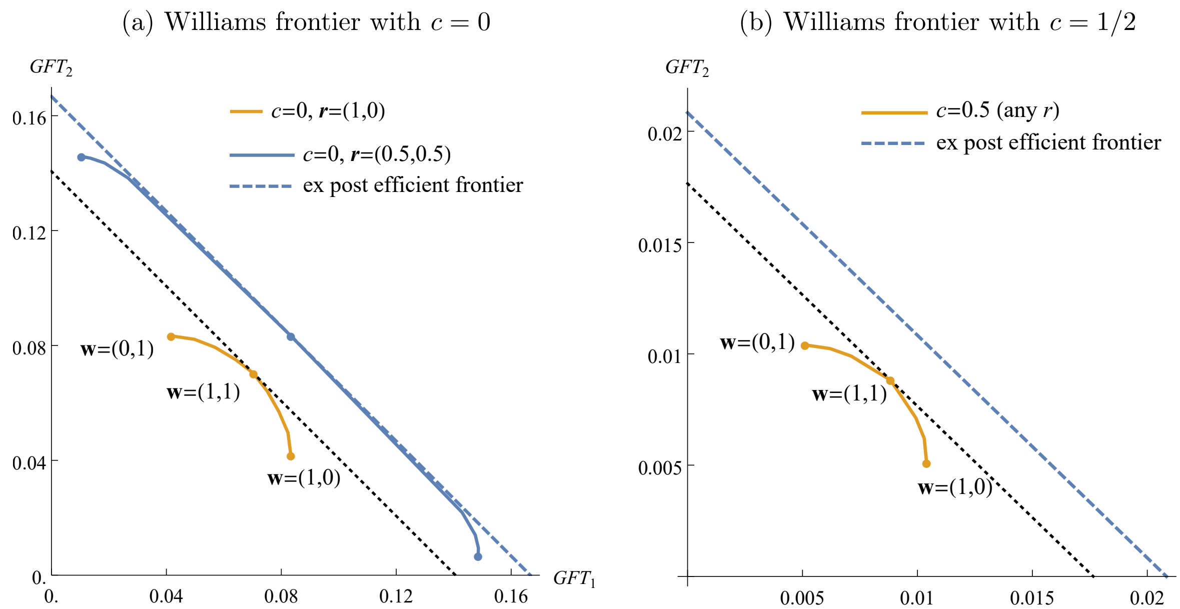

Figure 7: Williams frontier. Axes are the expected gains from trade, $GFT_i \equiv \mathbb{E}_{\mathbf{v}}[Q_i(\mathbf{v})v_i - M_i(\mathbf{v}) - r_i v_i]$ for the mechanism that maximizes weighted expected social surplus with weights $\mathbf{w}$. Assumes $n = 2$ and uniformly distributed types.

[27] In this case, as for the constrained efficient problem without bargaining weights, we again have a saddle point problem involving maximization with respect to the allocation rule and minimization with respect to the agents' worst-off types. We can rewrite the Lagrangian associated with the maximization problem as

$$\rho \mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^{n} \Psi_{i,\frac{w_i}{\rho}}(v_i;\omega_i) Q_i(\mathbf{v}) - T_{\mathbf{r},C}(\mathbf{Q}(\mathbf{v})) \right] + \sum_{i=1}^{n} r(w_i - \rho)_i \omega_i + \sum_{i=1}^{n} (w_i - \rho + \mu_i)\, u_i(\omega_i),$$

where the weight on the virtual type function for agent i becomes w_i/ρ rather than $1/\rho$ as we had in the unweighted version. It follows that we can extend $\hat{V}_{ij}^{ce}(\mathbf{v}, C; \rho, \boldsymbol{\omega})$ to incorporate bargaining weights, $\hat{V}_{ij}^{ce}(\mathbf{v}, C; \rho, \boldsymbol{\omega}, \mathbf{w})$, simply by replacing the weights in the ironed weighted virtual type functions. That is, replace $\overline{\Psi}_{i,\frac{1}{\rho}}(v_i;\omega_i)$ with $\overline{\Psi}_{i,\frac{w_i}{\rho}}(v_i;\omega_i)$. In this case, the solution value for the Lagrange multiplier on the no-deficit constraint satisfies $\rho^* \geq \max \mathbf{w}$.

6 Conclusions

We study trade on networks with linear transportation costs. We show that absent individual rational and no-deficit constraints an extremal ownership structure, that is, giving to entire resource to a specific single agent, is *always* optimal for the planner, irrespective of agents' type distributions, the transportation cost, and the network structure. For incomplete networks such as the star with three or more agents or the wheel with five or more agents, placing the entire resource at the hub is uniquely optimal with identical distributions, provided that transportation cost is positive but not prohibitively large. However, as we show, if resource holdings bestow an agent with property rights over the resource, then an extremal ownership structure conflicts with individual rationality and no-deficit constraints. We then solve for optimal ownership structure, anticipating that a constrained efficient rather than ex post efficient mechanism may be required for the trade once agents' types are realized. Numerical results for the uniform distribution and the star network show that the optimal ownership structure becomes more centralized the larger is the number of agents and the larger are the transportation costs. Even though the planner's optimal ownership structure is the same for the star and the wheel networks, the ownership structures that account for individual rationality and no-deficit constraints differ because the wheel offers additional opportunities of trade along the ring road.

One natural way to interpret these results is that the solution to the ownership structure problem faced by a social planner that does not have to respect the agents' individual rationality constraints is completely centralized for the star and wheel networks. Importantly, centralization occurs not because the planner cares for power, but because it achieves the first-best. In contrast, a "market" economy in which resource holdings confer property rights and where the market mechanism maximizes expected social surplus subject to incentive compatibility, individual rationality and no-deficit constraints, is characterized by an optimal ownership structure that is less centralized (and less efficient). Alternatively, the planner's problem can be interpreted as the problem an integrated firm solves that does not have to satisfy the individual rationality constraints of its agents. Viewed in this way, the paper provides a new channel for merger synergies—in the face of transportation costs, efficient resource allocation can only become possible within an integrated firm. For two agents with uniformly distributed types, we show that transportation cost problem can be so severe that ex post efficient trade is impossible for *any* ownership structure.

Our analysis also sheds new light on the long-standing question of why planned economies tend to centralize resources, whereas under private ownership, resources are more decentralized. If the central authority does not have to respect the agents' individual rationality constraints, for example, because they are part of an integrated firm from which walking away is costly or because it is the state that can use coercion, then the optimal resource allocation is completely centralized in, say, star and wheel networks. In contrast, it is decentralized in the designer's problem for small enough transportation costs. Centralization in the planner's problem here arises not because the planner wants somehow to accumulate power or influence, but because it is first-best. In a similar vein, because the first-best is possible in the planner's problem but not, in general, in the designer's, the paper also uncovers resource placement and costly transportation as a channel for synergies from the integration of firms that does not depend on contractual restrictions, viewing the planner's problem as that faced by an integrated firm and the designer's solution as the best that the market can achieve when all firms are independent.

An interesting question for future research is that once one has solved all the problems on the network, one can then ask what is the optimal network, assuming that forming links comes at some cost and anticipating the solutions on the network.

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A Proofs

Proof of Proposition 1. Letting $\hat{v}_{ji}(C) = \mathbb{E}_{\mathbf{v}}[\hat{V}_{ji}(\mathbf{v}, C)]$, we have $t^e_C(\mathbf{r}) = \sum_{i=1}^n \sum_{j=1}^n C_{ji}\hat{v}_{ji}(C)r_j$ and $ss^e_C(\mathbf{r}) = \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^n \sum_{j=1}^n v_i\hat{V}_{ji}(\mathbf{v}, C)r_j\right] - t^e_C(\mathbf{r})$. Using $r_n = 1 - \sum_{\ell=1}^{n-1} r_\ell$, we have

$$\begin{aligned} ss^e_C(\mathbf{r}) &= \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^n \sum_{j=1}^{n-1} v_i\hat{V}_{ji}(\mathbf{v}, C)r_j + \sum_{i=1}^n v_i\hat{V}_{ni}(\mathbf{v}, C)\left(1 - \sum_{\ell=1}^{n-1} r_\ell\right)\right] \\ &\quad - \sum_{i=1}^n \sum_{j=1}^{n-1} C_{ji}\hat{v}_{ji}(C)r_j - \sum_{i=1}^n C_{ni}\hat{v}_{ni}(C)\left(1 - \sum_{\ell=1}^{n-1} r_\ell\right), \end{aligned}$$

so for $j \in \{1, \ldots, n-1\}$, we have

$$\frac{\partial ss^e_C(\mathbf{r})}{\partial r_j} = \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^n v_i\left(\hat{V}_{ji}(\mathbf{v}, C) - \hat{V}_{ni}(\mathbf{v}, C)\right)\right] - \sum_{i=1}^n \left(C_{ji}\hat{v}_{ji}(C) - C_{ni}\hat{v}_{ni}(C)\right),$$

which is independent of r_j (and any other r_i). This implies that an extremal ownership structure is ***always*** optimal, independently of network structure and distributions. ■

Proof of Proposition 2. For a complete network (and identical distributions), any ownership structure $\mathbf{r}$ maximizes expected social surplus under the ex post efficient trade rule because any agent is ex ante interchangeable with any other. More precisely, for a complete network, expected social surplus is

$$\begin{aligned} &\sum_{i\in\mathcal{N}} \mathbb{E}_{\mathbf{v}}[v_i r_i \mid \max \mathbf{v} - c < v_i] \Pr(\max \mathbf{v} - c < v_i) \\ +\ &\sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}\setminus\{i\}} \mathbb{E}_{\mathbf{v}}[(v_i - c)r_j \mid v_i = \max \mathbf{v} > v_j + c] \Pr(v_i = \max \mathbf{v} > v_j + c). \end{aligned}$$

To see this, note that agent i retains its resource r_i whenever $\max \mathbf{v} - c < v_i$ (ignoring ties for the maximum value, which are a zero probability event) because then it is not optimal to transfer agent i's resources to any other agent. And, agent i consumes agent $j \neq i$'s resource r_j whenever $v_i = \max \mathbf{v}$ (again ignoring ties) and $v_i > v_j + c$, but at a transportation cost of

cr_j. Letting $r_n \equiv 1 - \sum_{i=1}^{n-1} r_i$, the derivative with respect to r_ℓ for $\ell < n$ is

$$\begin{aligned}
& \mathbb{E}_{\mathbf{v}}[v_\ell \mid \max \mathbf{v} - c < v_\ell] \Pr(\max \mathbf{v} - c < v_\ell) \\
+ & \sum_{i \neq \ell} \mathbb{E}_{\mathbf{v}}[(v_i - c) \mid v_i = \max \mathbf{v} > v_\ell + c] \Pr(v_i = \max \mathbf{v} > v_\ell + c) \\
- & \mathbb{E}_{\mathbf{v}}[v_n \mid \max \mathbf{v} - c < v_n] \Pr(\max \mathbf{v} - c < v_n) \\
- & \sum_{i \neq n} \mathbb{E}_{\mathbf{v}}[(v_i - c) \mid v_i = \max \mathbf{v} > v_n + c] \Pr(v_i = \max \mathbf{v} > v_n + c) \\
= & \ 0,
\end{aligned}$$

where the final equality holds by the assumption of identical distributions. Thus, for all $\mathbf{r}$, the derivative of expected social surplus with respect to any r_i is zero, and so the ownership structure is irrelevant for expected social surplus. In particular, maximum expected social surplus is achieved by placing all resources with a single agent.

To obtain another network structure, we remove links from the complete network, which weakly decreases expected social surplus, but by placing all resources with a completely connected agent, if one exists, we achieve the same expected social surplus as in the complete network with resources placed at that agent, which maximizes expected social surplus. If resources are placed only with agents in $\mathcal{I}$ in the sense that $\sum_{i \in \mathcal{I}} r_i = 1$, then the derivatives with respect to r_i for $i \in \mathcal{I}$ are zero, and the derivatives with respect to r_j for $j \notin \mathcal{I}$ are negative. For $c \in (0, 1)$, this ownership structure is uniquely optimal because it allows trades that would occur under the complete network and that would not occur with an alternative ownership structure. ■

Proof of Proposition 4. Suppose $c \geq 1/2$, in which case agents only ever trade with their immediate neighbors. Let $\mathcal{H}_i \subset \mathcal{N}$ be the set of agent i's immediate neighbors. Consider the mechanism in which agent i can buy from agent $j \in \mathcal{H}_i$ at (per-unit) price $\max\{v_j + c, \max_{h \in \mathcal{H}_j} v_h\}$ and agent i can sell to agent $j \in \mathcal{H}_i$ at (per-unit) price $v_j - c$. This mechanism induces agent i to demand r_j units from agent j if $v_i > \max\{v_j + c, \max_{h \in \mathcal{H}_j} v_h\}$, and zero units otherwise, and it induces agent i to offer r_i units to agent j if $v_i < v_j - c$ and $v_j = \max_{\ell \in \mathcal{H}_i} v_\ell$, and zero units otherwise. Thus, this mechanism induces the ex post efficient trade with trading buyers paying the lowest Walrasian price and trading sellers receiving the highest Walrasian price. Agents with type 1/2 do not trade and have zero payments. These types are worst-off, implying that worst-off types satisfy ex post and interim individual rationality constraints with equality. Turning to the budget surplus of this mechanism, if $v_i > \max\{v_j + c, \max_{h \in \mathcal{H}_j} v_h\}$, then agent i purchases r_j units from agent j and makes a payment of $r_j \max\{v_j + c, \max_{h\in\mathcal{H}_j} v_h\}$, while agent j is paid $r_j(v_i - c)$. Thus, budget surplus associated with trades involving agent i is

$$\begin{aligned}
&\sum_{\ell\in H_i \text{ s.t. } v_i>\max\{v_\ell+c,\max_{h\in\mathcal{H}_\ell} v_h\}} \left(r_\ell \max\{v_\ell + c, \max_{h\in\mathcal{H}_\ell} v_h\} - r_\ell (v_i - c)\right) \\
= &\sum_{\ell\in H_i \text{ s.t. } v_i>\max\{v_\ell+c,\max_{h\in\mathcal{H}_\ell} v_h\}} r_\ell \left(\max\{v_\ell - v_i + 2c, \max_{h\in\mathcal{H}_\ell} v_h - v_i + c\}\right),
\end{aligned}$$

where $v_\ell - v_i + 2c = (v_\ell + c) - v_i + c < v_i - v_i + c = c$ and $\max_{h\in\mathcal{H}_\ell} v_h - v_i + c < v_i - v_i + c = c$, which says that the transportation costs are not covered (on a trade-by-trade basis). This completes the proof of the impossibility of ex post efficient trade. ■

Proof of Lemma 2. Define

$$u_i(v) \equiv q_i(v)v - m_i(v) - r_i v,$$

implying that the individual rationality condition can be stated as for all $v \in [\underline{v}, \overline{v}]$, $u_i(v) \geq 0$. By incentive compatibility, $u_i(v) = \max_{\hat{v}} q_i(\hat{v})v - m_i(\hat{v}) - r_i v$, which implies that u_i is differentiable almost everywhere and by the envelope theorem, whenever it is differentiable, we have

$$u_i'(v) = q_i(v) - r_i.$$

Thus, for all $\omega \in [\underline{v}, \overline{v}]$,

$$u_i(v) = \int_\omega^v (q(x) - r)dx + u_i(\omega).$$

From this, it follows that

$$m_i(v) = q_i(v)v - r_i v - \int_\omega^v (q(x) - r)dx - u_i(\omega)$$

and so

$$\begin{aligned}
\mathbb{E}_{v_i}[m_i(v_i)] &= \int_{\underline{v}}^{\overline{v}} (q_i(x) - r_i)\, x dF_i(x) - \int_{\underline{v}}^{\overline{v}} \int_\omega^y (q(x) - r) f_i(y) dx dy - u_i(\omega) \\
&= \int_{\underline{v}}^{\overline{v}} (q_i(x) - r_i)\, \Psi_i(x;\omega) dF_i(x) - u_i(\omega) \\
&= \int_{\underline{v}}^{\overline{v}} q_i(x)\Psi_i(x;\omega) dF_i(x) - r_i\omega - u_i(\omega) \\
&= \mathbb{E}_{v_i}[q_i(v_i)\Psi_i(v_i;\omega)] - r_i\omega - u_i(\omega).
\end{aligned}$$

■

Proof of Proposition 6. Given c that permits ex post efficient allocation, consider the ownership structure $\tilde{\mathbf{r}} \in \mathcal{R}(c)$ that maximizes the budget surplus before fixed payments to the agents under the ex post efficient allocation rule. Letting $t^e(\mathbf{r}_{-n})$ denote the expected transportation cost under the ex post efficient allocation given $\mathbf{r}_{-n}$ and $r_n = 1-\sum_{i=1}^{n-1} r_i$, it follows that $\tilde{\mathbf{r}}_{-n}$ satisfies

$$\begin{aligned} \tilde{\mathbf{r}}_{-n} \quad \in \quad & \arg\max_{\mathbf{r}_{-n}} \mathbb{E}_{\mathbf{v}}[\sum_{i=1}^{n-1} \Psi(v_i;\omega_i^e(r_i))q_i^e(v_i) + \Psi(v_n;\omega_n^e(1-\sum_{j=1}^{n-1} r_j))q_n^e(v_n)] \\ & -\sum_{i=1}^{n-1} \omega_i^e(r_i)r_i - \omega_n^e(1-\sum_{j=1}^{n-1} r_j)(1-\sum_{j=1}^{n-1} r_j) - t^e(\mathbf{r}_{-n}), \end{aligned}$$

where $\omega_i^e(r_i) = (q_i^e)^{-1}(r_i)$. Note that

$$\mathbb{E}_{v_i}[\Psi(v_i;\omega_i^e(r_i))q_i^e(v_i)] = \int_0^{\omega_i^e(r_i)} \Psi^S(x)q_i^e(x)dF(x) + \int_{\omega_i^e(r_i)}^1 \Psi^B(x)q_i^e(x)dF(x),$$

and which has derivative with respect to r_i of

$$\begin{aligned} \frac{d}{dr_i}\mathbb{E}_{v_i}[\Psi(v_i;\omega_i^e(r_i))q_i^e(v_i)] \quad &= \quad \Psi^S(\omega_i^e(r_i))q_i^e(\omega_i^e(r_i))f(\omega_i^e(r_i))\omega_i^{e\prime}(r_i) \\ & \quad\quad -\Psi^B(\omega_i^e(r_i))q_i^e(\omega_i^e(r_i))f(\omega_i^e(r_i))\omega_i^{e\prime}(r_i) \\ &= \quad \left(\Psi^S(\omega_i^e(r_i)) - \Psi^B(\omega_i^e(r_i))\right) r_i f(\omega_i^e(r_i))\omega_i^{e\prime}(r_i) \\ &= \quad r_i\omega_i^{e\prime}(r_i), \end{aligned}$$

and note that for $i \in \{1,\ldots,n-1\}$,

$$\frac{\partial t^e(\mathbf{r}_{-n})}{\partial r_i} \equiv \mathbb{E}_{\mathbf{v}}[\sum_{j=1}^{n} C_{ij}\hat{V}_{ij}(\mathbf{v},C) - \sum_{j=1}^{n} C_{nj}\hat{V}_{nj}(\mathbf{v},C)].$$

Thus, we have the following first-order condition for $i \in \{1,\ldots,n-1\}$:

$$\begin{aligned} \omega_i^e(r_i) \quad &= \quad \omega_n^e(1-\sum_{j=1}^{n-1} r_j) - \frac{\partial t^e(\mathbf{r}_{-n})}{\partial r_i} \\ &= \quad \omega_n^e(1-\sum_{j=1}^{n-1} r_j) - \mathbb{E}_{\mathbf{v}}[\sum_{j=1}^{n} C_{ij}\hat{V}_{ij}(\mathbf{v},C) - \sum_{j=1}^{n} C_{nj}\hat{V}_{nj}(\mathbf{v},C)], \end{aligned}$$

which completes the proof. ■

Proof of Lemma 3. Let $q_i(v)$ be agent i's interim expected allocation when of type v and let r_i be its initial resources. For $v \geq 1-c$, we have $q_i(v) \geq r_i$ because i cannot never act as a seller. For $v \leq c$, we have $q_i(v) \leq r_i$ because i cannot never act as a buyer. This implies that for $c > 1/2$, in which case we have $1-c \leq c$, the agent's interim expected allocation satisfies $q_i(v) = r_i$ for all $v \in [1-c, c]$. Hence, for $c \geq 1/2$, all types $v \in [1-c, c]$ will be worst-off. ■

Proof of Lemma 4. Assume that $F_i = F$ for all $i \in \mathcal{N}$. Define the function $\hat{z}(\hat{v}, a)$ to be the implicit solution for the ironing parameter z that solves (8) (this is the same for all i given the assumption that $F_i = F$ for all $i \in \mathcal{N}$). When $c \geq 1/2$, ex post efficient trade is not possible by Proposition 4, so we have $\rho^* > 1$. Thus, $\hat{z}(1/2, 1/\rho^*) \in (0,1)$, and we can let $\hat{c} \in [1/2, 1)$ be such that for all $c \geq \hat{c}$, we have

$$1 - c < \hat{z}(1/2, 1/\rho^*) < c. \tag{A.1}$$

Focusing on the expression in (9) in square brackets, if $1 - c < z(1/\rho^*, \omega^*) \equiv z^* < c$, then in order to have $\hat{V}_{ij} = 1$ for $i \neq j$, we require that $\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) < z^*$, which implies that $\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) = \Psi_{\frac{1}{\rho^*}}(v_i; \omega_i^*)$, and $\overline{\Psi}_{\frac{1}{\rho^*}}(v_j; \omega_j^*) > z^*$, which implies that $\overline{\Psi}_{\frac{1}{\rho^*}}(v_j; \omega_j^*) = \Psi_{\frac{1}{\rho^*}}(v_j; \omega_j^*)$. So the term in square brackets can be written as

$$\sum_{i=1}^{n} \sum_{j \in \mathcal{N} \setminus \{i\}} \left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) - C_{ji} \right) \hat{V}_{ji}^{ce}(\mathbf{v}, C; \rho^*, \boldsymbol{\omega}^*) r_j + \sum_{i=1}^{n} \Psi_{\frac{1}{\rho^*}}(v_i; \omega_i^*) \hat{V}_{ii}^{ce}(\mathbf{v}, C; \rho^*, \boldsymbol{\omega}^*) r_i.$$

But notice that, dropping the arguments on $\hat{V}^{ce}$,

$$\begin{aligned}
& \mathbb{E}_{\mathbf{v}} \left[\left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) - \Psi_{\frac{1}{\rho^*}}(v_i; \omega_i^*) \right) \hat{V}_{ii}^{ce} \right] \\
= \; & \mathbb{E}_{\mathbf{v}} \left[\left(z^* - \Psi_{\frac{1}{\rho^*}}(v_i; \omega_i^*) \right) \hat{V}_{ii}^{ce} \mid \overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) = z^* \right] \Pr\left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) = z^* \right) \\
= \; & \mathbb{E}_{\mathbf{v}} \left[z^* - \Psi_{\frac{1}{\rho^*}}(v_i; \omega_i^*) \mid \overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) = z^*, \, \hat{V}_{ii}^{ce} = 1 \right] \Pr\left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) = z^*, \, \hat{V}_{ii}^{ce} = 1 \right) \\
= \; & \mathbb{E}_{\mathbf{v}} \left[z^* - \Psi_{\frac{1}{\rho^*}}(v_i; \omega_i^*) \mid \overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) = z^* \right] \Pr\left(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i; \omega_i^*) = z^* \right) \\
= \; & 0,
\end{aligned}$$

where the first equality uses the fact that the ironed and unironed virtual types are identical outside of the ironing range, the second equality uses the binary nature of $\hat{V}^{ce}$, the third equality uses the result that if v_i is in the ironing range and $c > \hat{c}$, then it is not possible for agent i to trade and so $\hat{V}_{ii}^{ce} = 1$, and the final equality uses the definition of the ironing parameter given in (8). Thus, the expectation of the expression in (9) in square brackets is

$$\mathbb{E}_{\mathbf{v}}\Big[\sum_{j=1}^{n}\sum_{i=1}^{n}\Big(\overline{\Psi}_{\frac{1}{\rho^*}}(v_i;\omega_i^*) - c_{ji}\Big)\,\hat{V}_{ji}^{ce}(\mathbf{v}, C;\rho^*, \boldsymbol{\omega}^*)r_j\Big],$$

which completes the proof. ■

Proof of Lemma 5. In the n star with $v_1 = 0$, expected social surplus under the ex post efficient allocation rule is

$$\begin{aligned}
& \mathbb{E}_{\mathbf{v}}\left[\frac{1-r}{n-1}\sum_{i=2}^{n} v_i \mid v_{(1)} \leq c\right]\Pr(v_{(1)} \leq c) \\
& + \mathbb{E}_{\mathbf{v}}\left[\frac{1-r}{n-1}\sum_{i=2}^{n} v_i + \sum_{j=2}^{n}\max\{0, v_{(1)} - 2c - v_j\}\frac{1-r}{n-1} + (v_{(1)} - c)r \mid v_{(1)} > c\right]\Pr(v_{(1)} > c) \\
= \; & (1-r)\mathbb{E}_{v_i}[v_i] + \mathbb{E}_{\mathbf{v}}\left[\max\{0, v_{(1)} - c)r\}\right] + \mathbb{E}_{\mathbf{v}}\left[\sum_{j=2}^{n}\max\{0, v_{(1)} - v_j - 2c\}\frac{1-r}{n-1}\right] \\
= \; & (1-r)\mathbb{E}_{v_i}[v_i] + \mathbb{E}_{\mathbf{v}}\left[\max\{0, v_{(1)} - c)r\}\right] + (1-r)\mathbb{E}_{\mathbf{v}}\left[\max\{0, v_{(1)} - v_n - 2c\}\right],
\end{aligned}$$

where the first line reflects the case in which all spokes have a value less than or equal to c and so the hub retains its initial resources of r, and the second line reflects the case in which at least one of the spokes has value greater than c, and so the spoke with the maximum value consumes its initial resource holding as well as the hub's holding of r and the holdings $\frac{1-r}{n-1}$ at other spokes with values less than $v_{(1)} - 2c$, while spokes with values in $[v_{(1)} - 2c, v_{(1)})$ consume their initial resource holding of $\frac{1-r}{n-1}$.

Expected social surplus under the ex post efficient allocation rule is linear in r, and so the optimum is extreme, with either $r = 0$ or $r = 1$. If $r = 0$, then the expected social surplus is

$$\mathbb{E}_{v_i}[v_i] + \mathbb{E}_{\mathbf{v}}\left[\max\{0, v_{(1)} - v_j - 2c\}\right],$$

and if $r = 1$, then the expected social surplus is

$$\mathbb{E}_{\mathbf{v}}\left[\max\{0, v_{(1)} - c)\}\right].$$

Thus, the expected social surplus with $r = 0$ is greater than or equal to that with $r = 1$ if and only if inequality (10) holds. ■

Proof of Proposition 15. The proof of the first statement in the proposition follows from evaluating (10) at $n = 2$ and from taking the limit of the right side (10) as n goes to infinity and noting that the inequality is not satisfied for c positive but sufficiently close to zero.

For $c \geq 1/2$, we can rewrite inequality (10) as $\mathbb{E}_{v_i}[v_i] \geq \mathbb{E}_{\mathbf{v}}\left[\max\{0, v_{(1)} - c\}\right]$, which we can rewrite as $\int_0^1 x dF(x) \geq \int_c^1 (y-c) dF^{n-1}(y)$ or, integrating by parts and rearranging, as

$$\frac{1}{2} \geq \frac{1}{2} + \int_0^1 F(x)dx - c - \int_c^1 F^{n-1}(y)dy. \tag{A.2}$$

The derivative of the right side with respect to c is $-1 + F^{n-1}(c)$, which is negative for $c < 1$. Thus, the right side of inequality (A.2) is decreasing in c for $c \in [1/2, 1)$. So a sufficient condition for inequality (A.2) to hold for all $c \in [1/2, 1]$ is that inequality (A.2) holds at $c = 1/2$, i.e., that

$$\frac{1}{2} \geq \int_0^1 F(x)dx - \int_{1/2}^1 F^{n-1}(y)dy. \tag{A.3}$$

Taking the derivative of the right side of inequality (A.3) with respect to n, we get $-\int_{1/2}^1 \ln(F(y))F^{n-1}(y)dy > 0$, and taking the limit of the right side with respect to n, we get $\lim_{n\to\infty} \int_0^1 F(x)dx - \int_{1/2}^1 F^{n-1}(y)dy = \int_0^1 F(x)dx$. So, we conclude that inequality (A.3) holds for all $c \in [1/2, 1]$ and $n \geq 2$ if $1/2 \geq \int_0^1 F(x)dx = 1 - \mathbb{E}_v[v]$. This completes the proof that the planner optimally has $r = 0$ in this case.

In contrast, if $1/2 < \int_0^1 F(x)dx = 1 - \mathbb{E}_v[v]$, then for a given $\hat{c} \in [1/2, 1 - \mathbb{E}_v[v])$, taking the limit of the right side of inequality (A.2) as n goes to ∞, we have

$$\lim_{n\to\infty} \frac{1}{2} + \int_0^1 F(x)dx - \hat{c} - \int_{\hat{c}}^1 F^{n-1}(y)dy = \frac{1}{2} + \int_0^1 F(x)dx - \hat{c} > \frac{1}{2},$$

implying that inequality (A.2) does not hold, and so the planner optimally has $r = 1$ in this case. ■

Proof of Proposition 16. We begin by showing that $\Pr(v_1 \geq v_2 + c) - \Pr(v_2 \geq v_1 + c) > 0$. We have

$$\begin{aligned}
\Pr(v_1 \geq v_2 + c) - \Pr(v_2 \geq v_1 + c) &= \int_c^1 \int_0^{v_1 - c} dG(v_2)dF(v_1) - \int_c^1 \int_0^{v_2 - c} dF(v_1)dG(v_2) \\
&= \int_c^1 \int_0^{y-c} dG(x)dF(y) - \int_c^1 \int_0^{y-c} dF(x)dG(y) \\
&= \int_c^1 \left[\frac{G(x-c)}{g(x)} - \frac{F(x-c)}{f(x)}\right] g(x)f(x)dx.
\end{aligned}$$

We now show that under MLRP, $\frac{G(x-c)}{g(x)} - \frac{F(x-c)}{f(x)} > 0$. To see this, use the rearranged version of the MLRP condition to obtain

$$\int_0^{x-c} f(x)g(y)dy = f(x)G(x-c) > g(x)F(x-c) = \int_0^{x-c} f(y)g(x)dy,$$

which completes the first part of the proof.

Now we show that under MLRP,

$$\mathbb{E}_{\mathbf{v}}[v_1 \mid v_2 - c \leq v_1 \leq v_2 + c] - \mathbb{E}_{\mathbf{v}}[v_2 \mid v_1 - c \leq v_2 \leq v_1 + c] > 0$$

holds for $c > 0$ sufficiently small. We have

$$\begin{aligned} &(\mathbb{E}[v_1 \mid v_2 - c \leq v_1 \leq v_2 + c] - \mathbb{E}[v_2 \mid v_1 - c \leq v_2 \leq v_1 + c]) \Pr(v_2 - c \leq v_1 \leq v_2 + c) \\ &= \int_0^{1-c} \int_{v_2}^{v_2+c} (v_1 - v_2) dF(v_1) dG(v_2) + \int_{1-c}^{1} \int_{v_2}^{1} (v_1 - v_2) dF(v_1) dG(v_2) \\ &+ \int_0^{1-c} \int_{v_1}^{v_1+c} (v_1 - v_2) dG(v_2) dF(v_1) + \int_{1-c}^{1} \int_{v_1}^{1} (v_1 - v_2) dG(v_2) dF(v_1). \end{aligned}$$

Letting

$$\begin{aligned} A &\equiv \int_0^{1-c} \int_{v_2}^{v_2+c} (v_1 - v_2) dF(v_1) dG(v_2) \\ B &\equiv \int_{1-c}^{1} \int_{v_2}^{1} (v_1 - v_2) dF(v_1) dG(v_2) \\ C &\equiv \int_0^{1-c} \int_{v_1}^{v_1+c} (v_1 - v_2) dG(v_2) dF(v_1) \\ D &\equiv \int_{1-c}^{1} \int_{v_1}^{1} (v_1 - v_2) dG(v_2) dF(v_1), \end{aligned}$$

we have

$$B = \int_{1-c}^{1} (1 - F(x))G(x)dx \quad \text{and} \quad D = -\int_{1-c}^{1} (1 - G(x))F(x)dx$$

and thus

$$B + D = \int_{1-c}^{1} \left((1 - F(x))G(x) - (1 - G(x))F(x)\right) dx > 0,$$

where the inequality follows because MLRP implies FOSD, i.e., $1 - F \geq 1 - G$ and $G \geq F$, and thus $(1 - F)G \geq (1 - G)F$.

Integrating gives

$$A = \int_0^c yG(y)dF(y) + \int_c^1 y[G(y) - G(y-c)]dF(y) - \int_0^{1-c} y[F(y+c) - F(y)]dG(y)$$
$$C = -\int_0^c yF(y)dG(y) - \int_c^1 y[F(y) - F(y-c)]dG(y) + \int_0^{1-c} y[G(y+c) - G(y)]dF(y)$$

and thus

$$\begin{aligned} A + C = &\int_0^c y \left[\frac{G(y)}{g(y)} - \frac{F(y)}{f(y)} \right] f(y)g(y)dy \\ &+ \int_c^1 y \left\{ \left[\frac{G(y)}{g(y)} - \frac{F(y)}{f(y)} \right] + \left[\frac{G(y) - G(y-c)}{g(y)} - \frac{F(y) - F(y-c)}{f(y)} \right] \right\} f(y)g(y)dy \\ &+ \int_0^{1-c} y \left[\frac{G(y+c) - G(y)}{g(y)} - \frac{F(y+c) - F(y)}{f(y)} \right] f(y)g(y)dy. \end{aligned}$$

The result is thus established if one can show that $A + C \geq 0$.

Let us first rewrite $A + C$ as

$$\begin{aligned} A + C = &\int_0^1 y \left[\frac{G(y)}{g(y)} - \frac{F(y)}{f(y)} \right] f(y)g(y)dy \\ &+ \int_c^1 y \left[\frac{G(y) - G(y-c)}{g(y)} - \frac{F(y) - F(y-c)}{f(y)} \right] f(y)g(y)dy \\ &+ \int_0^{1-c} y \left[\frac{G(y+c) - G(y)}{g(y)} - \frac{F(y+c) - F(y)}{f(y)} \right] f(y)g(y)dy, \end{aligned}$$

where the first line and second line are positive and the third is negative. That the first line is positive follows from the implication of MLRP noticed above.

To see the other two results, use (12) for $x > y$ twice, first to obtain

$$\int_y^{y+c} f(x)g(y)dx = [F(y+c) - F(y)]\, g(y) > \int_y^{y+c} f(y)g(x)dx = [G(y+c) - G(y)]\, f(y),$$

which is to say that

$$\frac{G(y+c) - G(y)}{g(y)} - \frac{F(y+c) - F(y)}{f(y)} < 0,$$

and then to obtain

$$\int_{x-c}^{x} f(x)g(y)dy = [G(x) - G(x-c)]\, f(x) > \int_{x-c}^{x} f(y)g(x)dy = [F(x) - F(x-c)]\, g(x),$$

which is the same as

$$\frac{G(x)-G(x-c)}{g(x)}-\frac{F(x)-F(x-c)}{f(x)}>0.$$

Collecting terms, we get

$$\begin{aligned}A+C=&\int_{1-c}^{1} y\left[\frac{G(y)}{g(y)}-\frac{F(y)}{f(y)}\right] f(y)g(y)dy\\ &+\int_{c}^{1} y\left[\frac{G(y)-G(y-c)}{g(y)}-\frac{F(y)-F(y-c)}{f(y)}\right] f(y)g(y)dy\\ &+\int_{0}^{1-c} y\left[\frac{G(y+c)}{g(y)}-\frac{F(y+c)}{f(y)}\right] f(y)g(y)dy,\end{aligned}$$

where the first and a second line are still positive by MLRP. The last line is also positive for $c>0$ sufficiently small since by MLRP $\frac{G(y)}{g(y)}>\frac{F(y)}{f(y)}$ and the derivatives of $\frac{G(y+c)}{g(y)}$ and $\frac{F(y+c)}{f(y)}$ with respect to c, evaluated at $c=0$, are both 1. Thus, the inequality $\frac{G(y+c)}{g(y)}>\frac{F(y+c)}{f(y)}$ holds for all $c\in[0,\varepsilon)$ for some $\varepsilon>0$. ■

B Details for ordering of $\overline{r}_n(0)$

Assume that $F_i = F$ for all $i \in \mathcal{N}$. For general n and $c = 0$, we have $q^e(v) = H_n(v)$, where $H_n(v) \equiv F^{n-1}(v)$, so $\omega_i^e = H_n^{-1}(r_i)$. Using (7), $\bar{\mathbf{r}}(0)$ is defined by the $\mathbf{r}$ that solves

$$\mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n} \Psi(v_i; \omega_i^e) q^e(v_i)\right] = \sum_{i=1}^{n} \omega_i^e r_i,$$

which we can write as

$$\begin{aligned}
0 &= \mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n} \Psi(v_i; H_n^{-1}(r_i)) H_n(v_i)\right] - \sum_{i=1}^{n} H_n^{-1}(r_i) r_i \\
&= \sum_{i=1}^{n}\left(\int_0^{H_n^{-1}(r_i)} \Psi^S(x) H_n(x) f(x) dx + \int_{H_n^{-1}(r_i)}^{1} \Psi^B(x) H_n(x) f(x) dx - H_n^{-1}(r_i) r_i\right) \\
&= \sum_{i=1}^{n}\left(\int_0^{H_n^{-1}(r_i)} \left(x + \frac{F(x)}{f(x)}\right) H_n(x) f(x) dx + \int_{H_n^{-1}(r_i)}^{1} \left(x - \frac{1-F(x)}{f(x)}\right) H_n(x) f(x) dx - H_n^{-1}(r_i) r_i\right) \\
&= \sum_{i=1}^{n}\left(\int_0^{1} x H_n(x) f(x) dx + \int_0^1 H_n(x) F(x) dx - \int_{H_n^{-1}(r_i)}^{1} H_n(x) dx - H_n^{-1}(r_i) r_i\right) \\
&= 1 + (n-1)\int_0^1 F^n(x) dx - \sum_{i=1}^{n}\left(\int_{H_n^{-1}(r_i)}^{1} H_n(x) dx + H_n^{-1}(r_i) r_i\right) \\
&= 1 + (n-1)\int_0^1 F^n(x) dx - \sum_{i=1}^{n}\left(1 - \int_{H_n^{-1}(r_i)}^{1} x dH_n(x)\right) \\
&= (n-1)\left(\int_0^1 F^n(x) dx - 1 + \frac{1}{n-1}\sum_{i=1}^{n}\int_{H_n^{-1}(r_i)}^{1} x dH_n(x)\right).
\end{aligned}$$

Thus, for a star network with ownership structure $(r, \frac{1-r}{n-1}, \ldots, \frac{1-r}{n-1})$ and $c = 0$, we have

$$\int_0^1 F^n(x) dx = 1 - \frac{1}{n-1}\int_{(F^{n-1})^{-1}(r)}^{1} x dF^{n-1}(x) - \int_{(F^{n-1})^{-1}(\frac{1-r}{n-1})}^{1} x dF^{n-1}(x),$$

which for the uniform distribution can be written as

$$\frac{n}{n+1} = r^{\frac{n}{n-1}} + (n-1)^{\frac{-1}{n-1}} (1-r)^{\frac{n}{n-1}}.$$

Note that the left side is constant in r and the right side is convex in r and increasing in r at $r = 1$. Further, the right side is equal to 1 at $r = 1$, and so greater than the left side, and the right side is less than the left side at $r = 0$ for $n > 2$. So for all $n > 2$, there is a unique solution. Solving this (and taking the maximum solution for $n = 2$), we have:

n	$\overline{r}_n(0)$	$n\frac{1-r_1}{n-1}$
2	0.788675	0.433650
3	0.765431	0.351853
4	0.764689	0.313749
5	0.768943	0.288821
6	0.774292	0.270849
7	0.779633	0.257095
8	0.784635	0.246131
9	0.789221	0.237126
10	0.793397	0.229559

While $\overline{r}_n(0)$ is not monotone in n, that comparison does not properly account for the expansion in the number of agents. Here we show that the cumulative distributions $G_n(i) \equiv \sum_{\ell=1}^{i} r_\ell$ defined for $i \in \{0, 1, \ldots, n\}$ satisfy FOSD in n with G_n first-order stochastically dominating $G_{n'}$ if $n < n'$. More precisely, define the continuous functions

$$G_n(x) \equiv \begin{cases} nr_1 x & \text{if } x \leq 1/n, \\ 1 - \frac{1-r_1}{1-1/n} + \frac{1-r_1}{1-1/n} x & \text{if } x > 1/n. \end{cases}$$

The slope at $x = 1$, $\frac{1-r_1}{1-1/n}$, is decreasing in n, which is sufficient to prove FOSD.

C Details for the analysis with costly entry

In the entry game, the payoffs of the hub and spokes have the following analytic expressions for $c \in [0, 1/2]$ and $r \in [1/2, 1]$:

$$\pi_2^{hub}(c, r) = \frac{1}{6}(c^3(-2 + 3r) + (c^2 - c)(6 - 9r) + 2),$$

$$\pi_2^{spoke}(c, r) = \frac{1}{6}\left(c^3(1 - 3r) + (c^2 - c)(9r - 3) + 2\right),$$

$$\pi_3^{hub}(c,r) = \frac{1}{12}(c^4(-16+17r) + 28c^3(1-r) - 6c^2(2-r) - 4c(1-3r) + 3),$$

$$\pi_3^{spoke}(c,r) = \frac{1}{12}\left(c^4(8-9r) - 19c^3(1-r) + c^2(15-9r) - 4c(1+r) + 3\right),$$

$$\pi_4^{hub}(c,r) = \frac{1}{60}(c^5(-64+67r) + 140c^3(1-r) + 10c^2(-10+7r) + 45cr + 12),$$

$$\pi_4^{spoke}(c,r) = \frac{1}{180}(6c^5(8-9r) - 190c^3(1-r) - 30c^2(-7+5r) - 15c(5+r) + 36).$$

Table 3: Entry under the planner and designer. Assumes an $n = 4$ star, $c = 0.05$, and uniformly distributed types.

	planner		designer		
filled spokes	hub payoff	spoke payoff	$r_n^D(c)$	hub payoff	spoke payoff
0	0.5		1	0.5	
1	0.3571	0.2858	0.7708	0.3408	0.3021
2	0.2821	0.2179	0.7365	0.2687	0.2228
3	0.2363	0.1758	0.7250	0.2252	0.1775

Thus, depending upon entry costs, we get different numbers of spokes:

Table 4: Entry of spokes as a function of entry costs under the planner and designer. Assumes an $n = 4$ star, $c = 0.05$, and uniformly distributed types.

	number of spokes	
entry cost	planner	designer
$e < 0.1758$	3	3
$e \in (0.1758, 0.1775)$	2	3
$e \in (0.1775, 0.2179)$	2	2
$e \in (0.2179, 0.2228)$	1	2
$e \in (0.2228, 0.2858)$	1	1
$e \in (0.2858, 0.3021)$	0	1
$e > 0.3021$	0	0

D Appendix: Trade-sacrifice mechanism example

Consider a star network with $\mathbf{r} = (r, \frac{1-r}{n-1}, \ldots, \frac{1-r}{n-1})$ and $n = 3$. Let $SS^{TSM}(\mathbf{v}, c, r)$ denote social surplus.

For $c \in [1/2, 1]$, any trade is from a spoke to the hub, so we have

$$SS^{TSM}(\mathbf{v}, c, r) = \begin{cases} v_1 \frac{1+r}{2} - \frac{1-r}{2} c + \max\{v_2, v_3\} \frac{1-r}{2} & \text{if } \max\{v_2, v_3\} + c < v_1, \\ v_1 r + v_2 \frac{1-r}{2} + v_3 \frac{1-r}{2} & \text{otherwise.} \end{cases}$$

For $c \in (0, 1/2)$, we may also have trade from the hub to a spoke and from spoke to spoke, giving us social surplus for $c \in [0, 1/2)$ of

$$SS^{TSM}(\mathbf{v}, c, r) = \begin{cases} v_1 \frac{1+r}{2} - \frac{1-r}{2} c + \max\{v_2, v_3\} \frac{1-r}{2} & \text{if } \max\{v_2, v_3\} + c < v_1, \\ \max\{v_2, v_3\} \frac{1+r}{2} - rc + \min\{v_2, v_3\} \frac{1-r}{2} & \text{if } v_1 + c < \min\{v_2, v_3\} + 2c < \max\{v_2, v_3\}, \\ \max\{v_2, v_3\}(1-r) - \frac{1-r}{2} 2c + v_1 r & \text{if } \min\{v_2, v_3\} + 2c < v_1 + c < \max\{v_2, v_3\}, \\ v_1 r + v_2 \frac{1-r}{2} + v_3 \frac{1-r}{2} & \text{otherwise,} \end{cases}$$

where the first line is for trade from spoke to hub, the second for trade from hub to spoke, and the third for trade from spoke to spoke.[28]

Taking the expectation of this for uniformly distributed types, we get for $n = 3$,

$$\mathbb{E}_{\mathbf{v}}\left[SS^{TSM}(\mathbf{v}, c, r)\right] = \begin{cases} \frac{1}{12}(8 - 4c(3-r) + 6c^2(5-4r) - 4c^3(9-11r) + c^4(17-25r)) & \text{if } c \in [0, 1/2), \\ \frac{1}{12}(7 - r - 4c(1-r) + 6c^2(1-r) - 4c^3(1-r) + c^4(1-r)) & \text{if } c \in [1/2, 1]. \end{cases}$$

Thus, $\frac{\partial SS^{TSM}(c,r)}{\partial r} > 0$ for $c \in (0, 0.3218)$ and $\frac{\partial SS^{TSM}(c,r)}{\partial r} < 0$ for $c \in (0.328, 1)$.

[28] For $n = 3$ and $c = 0$, expected social surplus is

$$\mathbb{E}_{\mathbf{v}}\left[\frac{2}{3}\left(v_{(1)}\frac{1+r}{2} + v_{(2)}\frac{1-r}{2}\right) + \frac{1}{3}\left(v_{(1)}(1-r) + v_{(2)} r\right)\right] = \frac{2}{3},$$

where the first term inside the square brackets reflects cases in which there is trade between the hub and one spoke, in which case the highest valuing agent (whether hub or spoke) consumes $\frac{1+r}{2}$ and the nontrading spoke consumes $\frac{1-r}{2}$, and the second term inside the square brackets reflects cases in which there is trade between the two spokes, in which case the highest valuing agent (a spoke) consumes $1-r$ while the hub does not trade and so consumes r.

E Augmented trade-sacrifice mechanism

For $n > 2$, whether posting a (reserve) price p improves social surplus relative to the trade-sacrifice mechanism above depends on the details. For simplicity, we focus on star networks with $r_1 = 1$. Let $v_{(i:n-1)}$ be the i-th highest value of the $n-1$ independent draws on the spokes and consider the ***augmentation*** of the trade-sacrifice mechanism according to which trade occurs if and only if

$$v_{(1:n-1)} > \max\{p, v_{(2:n-1)}\} > v_1 + c.$$

If trade takes place, agent 1 sells r_1 to the agent with the highest value, who pays $\max\{p, v_{(2:n-1)}\}$ to agent 1. Relative to the mechanism without a reserve price p, augmentation increases social surplus if $v_{(1:n-1)} > p > v_1 + c > v_{(2:n-1)}$ and decreases it if $p > v_{(1:n-1)} > v_{(2:n-1)} > v_1 + c$.

Here we use v_i as shorthand for $v_{(i:n-1)}$ and use x as shorthand for v_1. Denote by $f_{12}(v_1, v_2) \equiv (n-1)(n-2)F^{n-3}(v_2)f(v_2)f(v_1)$ the joint density of the first and second-highest of $n-1$ draws from F and denote by $F_{(2:n-1)}(v) \equiv (n-1)(1-F(v))F^{n-2}(v) + F^{n-1}(v)$ the distribution of the second-highest of $n-1$ draws from F. We temporarily assume that the designer knows the distribution F. Then the optimal p maximizes

$$\begin{aligned}&\mathbb{E}_{\mathbf{v}}\left[v_{(1:n-1)} - c \mid v_1 + c < \max\{p, v_{(2:n-1)}\} < v_{(1:n-1)}\right] \Pr\left(v_1 + c < \max\{p, v_{(2:n-1)}\} < v_{(1:n-1)}\right)\\ &+\mathbb{E}_{v_1}\left[v_1 \mid v_1 + c \geq \max\{p, v_{(2:n-1)}\}\right] \Pr\left(v_1 + c \geq \max\{p, v_{(2:n-1)}\}\}\right)\end{aligned}$$

We can rewrite this expectation as:

$$\mathbb{E}_{\mathbf{v}}[SS] = \begin{cases} \int_c^1 \int_c^{v_1} (v_1 - c)\, F(v_2 - c) f_{12}(v_1, v_2) dv_2 dv_1 \\ \quad + \int_0^{1-c} x F_{(2:n-1)}(x+c) dF(x) + \int_{1-c}^1 x dF(x) \quad \text{if } p \leq c, \\ \\ \int_p^1 \int_0^{p-c} \int_0^p (v_1 - c)\, f(x) f_{12}(v_1, v_2) dv_2 dx dv_1 \\ \quad + \int_p^1 \int_p^{v_1} (v_1 - c)\, F(v_2 - c) f_{12}(v_1, v_2) dv_2 dv_1 \\ \quad + \int_{p-c}^{1-c} x F_{(2:n-1)}(x+c) dF(x) + \int_{1-c}^1 x dF(x) \quad \text{if } p > c. \end{cases}$$

Notice that $\mathbb{E}_{\mathbf{v}}[SS]$ is constant with respect to p for $p \leq c$ and that the right derivative of $\mathbb{E}_{\mathbf{v}}[SS]$ with respect to p at $p = c$ is $f(0) \int_c^1 \int_0^c (v_1 - c)\, f_{12}(v_1, v_2) dv_2 dv_1$, which is positive for $c \in (0, 1)$. Furthermore, focusing on the case of $c = 0$, the first nonzero right derivative is positive.[29] Thus, when $c = 0$, $\mathbb{E}_{\mathbf{v}}[SS]$ is also increasing in p at $p = c$. This implies that for all $c \in [0, 1)$, expected social surplus is increasing in p at $p = c$, and so the expected social surplus maximizing p is greater than c, giving us the following result:

Proposition E.1. *For all $c \in [0, 1)$, the p-augmented trade-sacrifice mechanism, with optimally chosen p given F, results in greater expected social surplus than the baseline mechanism.*

We illustrate the dependence on c and p of expected social surplus under the p-augmented tradesacrifice mechanism in Figure E.1(a) and the expected social surplus maximizing value for p, denoted by $p^*(c)$ in Figure E.1(b).

(a) Expected social surplus in the p-augmented tradesacrifice mechanism

(b) Expected social surplus maximizing p for the p-augmented trade-sacrifice mechanism

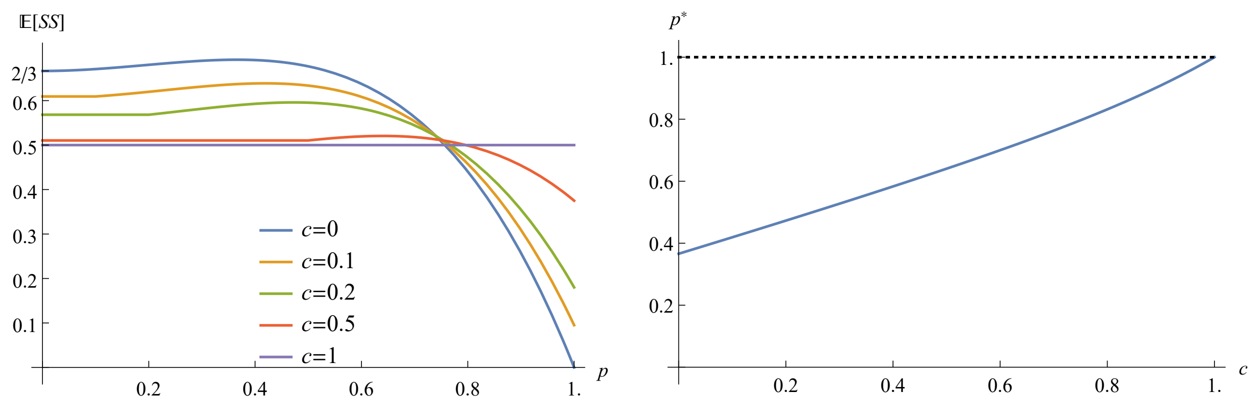

Figure E.1: Assumes a star network with $n = 3$, $\mathbf{r} = (1, 0, 0)$, and uniformly distributed types.

[29]The second right derivative of $\mathbb{E}_{\mathbf{v}}[SS]$ with respect to p at $p = c = 0$ is $f(0) \int_0^1 v_1 f_{12}(v_1, 0) dv_1$, which is positive for $n = 3$ and zero for larger n; and for $k \in \{3, 4, \dots\}$, the k-th right derivative of $\mathbb{E}_{\mathbf{v}}[SS]$ with respect to p at $p = c = 0$ is $f(0) \int_0^1 v_1 \frac{\partial^{k-2} f_{12}(v_1, 0)}{\partial v_2^{k-2}} dv_1$, which is positive for $n = k + 1$ and zero for larger n.