Optimal Prize Design in Team Contests with Pairwise Battles*

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Abstract

This paper studies the effort-maximizing design of a team contest with an arbitrary number (odd or even) of pairwise battles. In a setting with full heterogeneity across players and battles, the organizer determines the prize allocation rule (or the winning rule of an indivisible prize) contingent on battle outcomes. We propose a measure of team's strength, which plays a crucial role in prize design. The optimal design is a majority-score rule with a headstart score granted to the weaker team: All battles are assigned team-invariant scores, the weaker team is given an initial headstart score which is the difference in strengths between teams, and the team collecting higher total scores from its winning battles wins the entire prize. The optimal rule resembles the widely-adopted Elo rating system.

JEL Classification: C61, C72, D72, D74.

Keywords: Contest Design, Multi-battle Team Contest, Majority Rule, Headstart, Elo Rating, Linear Programming.

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1 Introduction

In many competitive circumstances, contenders from different teams compete in pairs on multiple disjoint fronts, and the winning team is determined by their overall performance over a series of battles. This type of team competition featuring pairwise battles can be found in R&D competitions, sporting events with team titles, political campaigns, and other competitive environments.[1] Fu, Lu, and Pan (2015) have conducted a thorough game-theoretical analysis of these team contests while assuming an exogenous majoritarian winning rule, i.e., a team wins the entire prize if it wins a majority of battles. In many of these competitions, a central question for the contest organizer, however, is how to appropriately design the prize allocation rule (or equivalently the winning rule if the prize is indivisible) to incentivize a more productive effort supply.[2] In this paper, we aim to answer this question by studying the effort-maximizing prize design in such team contests.[3]

The best-of-N winning rule (also referred to as the simple majority rule) is prevalent in such contests. For example, it is typically adopted in sporting events with team titles and the election for the House of Representatives between Republicans and Democrats (see, e.g., Snyder (1989), Klumpp and Polborn (2006)). This rule treats two opposing teams equally and allocates the entire prize to the team that wins the majority of battles. Apparently, it depends neither on the teams' identities nor on the order of wins.

Despite its popularity, it remains unclear whether this simple majority rule is most effective in inducing effort supply when a contest organizer has the freedom to set the prize structure. Generally, a prize allocation rule can be contingent on both the teams' identities and the full history of battle outcomes. Consider an R&D race between a local research

[1] For example, the National Natural Science Foundation of China (NSFC) funds tens of major projects every year, which are comprehensive and multidisciplinary in nature. In the call for proposals, a project specifies multiple areas that need to be investigated. A competing team usually consists of multiple parties from diverse universities or institutes. Each party specializes in preparing one research proposal in the area of their expertise.

[2] In sports context, the organizer typically values the effort supply of all players. In R&D competitions, every research team's effort contributes to generating innovative ideas for addressing the concerned issues. In political competitions, it is essential to incentivize all members from both parties to exert substantial efforts to maintain the operation of a healthy society.

[3] While Fu, Lu, and Pan (2015) adopt a majoritarian winning rule in their analysis, we accommodate a general class of prize allocation rules to identify the optimal prize design. Moreover, we generalize Fu, Lu, and Pan (2015) to settings with an arbitrary number of battles.

alliance and a foreign team. The foreign team usually has to outperform the local team by a sufficient margin to win the competition held by a local government. Moreover, the allocation rule can depend on the composition of winning battles rather than the number of winning battles, as in the US presidential election. Interesting questions thus arise: What does the optimal prize rule look like in general? Is there any theoretical rationale for adopting the majority rule beyond the justification of simplicity and fairness? How does the optimal design react to the degree of asymmetry between teams and the heterogeneity across battles?

To address these questions, we study the effort-maximizing prize allocation rules by granting a contest organizer full flexibility in rewarding each team based on the entire path of battle outcomes and the team's identity. We restrict our attention to the prize allocation rules that satisfy nonnegativity, monotonicity, and budget balance conditions, implying that the prizes are nonnegative, additional battle victory is never detrimental, and the prize budget is always wholly awarded. In our model, two teams with the same number (N) of players compete with each other. Each player from one team is exogenously matched to his counterpart from the rival team, and the matched pairs compete head-to-head on their own battlefields. The winner of each battle is determined through a winner-selection mechanism that exhibits homogeneity of degree zero in players' efforts (e.g., generalized Tullock contest). The team prize is a public good among its members, and each player chooses his own effort to maximize his payoff. Our study accommodates *full-fledged heterogeneity*: the contest technologies can differ across battles, and players can be completely heterogeneous within or across teams in their marginal effort costs.

We first formulate the contest organizer's effort-maximizing problem subject to the feasibility conditions of the prize structures. Under the budget balance condition, the history-independence result originally established by Fu, Lu, and Pan (2015) extends to our setting, which means that each battle can be viewed as independent lotteries with equilibrium winning probabilities irrelevant to the prize structure. Thus, the effective prize spread in each battle is a linear combination of prizes. The homogeneity of degree zero contest technology implies that each player's effort is proportional to the prize spread in each battle.[4] Moreover, nonnegativity, monotonicity, and budget balance conditions are all linear constraints, so the

[4]Please refer to Fu, Lu, and Pan (2015) for detail.

feasible prize allocation rules constitute a polytope. Thereby, both the total effort function and the constraints on the prize structures are linear in prizes.

The optimal prize design is established using an iterative adjustment method that consists of two steps. First, we show that the optimal design must be a vertex solution by applying the fundamental theorem of linear programming. This implies that the entire reward must be allocated to one team while the other team receives nothing, precluding the possibility of any intermediary rewards. Second, we iteratively eliminate the sub-optimal prize rules from the set of vertex rules, which renders the closed-form optimal rule. In this process, we discover an innovative measurement for assessing a team's strengths by aggregating the strengths of its members, which is crucial for identifying and interpreting the optimal design.

The optimal design takes a surprisingly simple and elegant form of a *majority-score rule with a headstart* score to the weaker team: All battles are assigned scores, which generally differ across battles. Both teams earn the same score for winning a battle. The battle score is proportional to the unbalancedness of the battle, weighted by the effectiveness in effort inducement.[5] The weaker team is endowed with an initial score to start, which equals the difference in team strengths. At the end of the game, the team collecting higher total scores wins the entire prize. The analysis is fully applicable when the designer maximizes the sum of weighted efforts across battles. One only needs to normalize players' marginal effort costs in each battle by the weight associated with the battle, and then apply the same procedure to pin down the optimal design.

We then proceed to a prominent special case, in which the winner-selection mechanism is uniform across battles, and players on each team are homogeneous. This setting only incorporates pure asymmetry between teams in terms of their players' marginal effort costs, which helps delineate the impact of the battle heterogeneity on the prize design. With homogeneous battles, all battles are assigned the same score, and a player's incentive depends only on his own ability and that of his opponent.[6] In this case, the optimal prize design is a path-independent rule named *majority rule with a headstart*, which allocates the entire prize

[5]The effectiveness is measured by the ratio of induced effort to prize spread; the unbalancedness of the battle is measured by the reciprocal of the product of players' winning chances.

[6]With full-fledged heterogeneity, battles, in general, carry different scores, and therefore a player's incentive for exerting effort also depends on the specific battle.

to the team winning a sufficient number of battles and favors the weaker team by awarding it a headstart in terms of an initial number of wins. Equivalently, the optimal rule rewards the entire prize to the stronger team when it wins at least $K_S(> N/2)$ battles; otherwise, the weaker team obtains the entire prize.

Our results indicate that the optimal design manages team asymmetry and battle heterogeneity completely through the headstart score and battle scores, respectively. Searching for the optimal design does not require going beyond a parsimonious class of score-based majority rules with a headstart. First, the designer uses a headstart to favor the underdog and handicap the favorite to level the playing field. Second, a higher battle score is assigned to a more productive battle to better incentivize its players. If all battles carry very close scores and two teams have similar levels of overall strengths, the widely adopted best-of-N rule is approximately optimal. These findings dramatically simplify the procedure to design the optimal prize rule and provide useful guidance on how to incentivize players in a pairwise team competition.

We further find that the optimal design can be alternatively interpreted from the perspective of the Elo rating system, which is broadly adopted by sports associations to rate players in bilateral games. By Elo rating, the winner of a game gains certain rating points from the loser, and the underdog can obtain more points than the favorite through a win. If a team's Elo rating change is measured as the sum of its players' Elo rating changes, our optimal design indicates that the team whose Elo rating improves takes the entire prize.[7]

Our paper primarily belongs to the literature on multi-battle contests, in which one branch focuses on contests between individuals while the other studies contests between teams. For the first branch, many studies discover strategic momentum/discouragement effect in dynamic individual contests, including Harris and Vickers (1987); Ferrall and Smith (1999); Klumpp and Polborn (2006); Konrad and Kovenock (2009); Gelder (2014); and Gauriot and Page (2019), among others. Other papers focus on prize designs in dynamic contests between individual players, including Feng and Lu (2018); Jiang (2018); Sela and Tsahi (2020); and Clark and Nilssen (2020), among others. In particular, Feng and Lu (2018) study the optimal contingent prize allocation in a sequential three-battle contest between two

[7]By construction, the sum of the two teams' Elo rating changes must be zero.

players. Lemus and Marshall (2022) provide empirical and experimental evidence showing that allowing contingent prizes can significantly improve contest outcomes in dynamic multi-battle contests.[8]

Our study aligns more closely with the literature on team contests. Fu, Lu, and Pan (2015) are the first to investigate team contests involving odd-length pairwise battles. Their research primarily focuses on equilibrium characterization under a simple majority rule and establishing related qualitative regularities and properties of the equilibrium including history independence.[9] Häfner (2017, 2022) analyzes tug-of-war contests between two teams, in which a team first accumulates n more battle victories than the other team wins the tug-of-war. Barbieri and Serena (2019) show that a simultaneous contest maximizes the winners' effort under the majority winning rule. Konishi, Pan, and Simeonov (2022) analyze equilibrium player ordering in majoritarian team contests. Differing from those studies, we endogenize the prize allocation rule by identifying the effort-maximizing prize structures while allowing full-fledged heterogeneity and an arbitrary number of battles. While setting up our design problem as a linear program crucially relies on the history independence result established by Fu, Lu, and Pan (2015), solving the problem and fully establishing the optimal design explicitly are far from trivial. We develop an elimination technique to establish a majority-score rule with a headstart as the optimal design, which generally differs from the simple majority rule.

Our paper is also closely related to the studies on single-battle group contests. Many of these assume that a group's win is a public good among its members, including Baik, Kim, and Na (2001); Barbieri, Malueg, and Topolyan (2014); Topolyan (2014), Chowdhury, Lee, and Topolyan (2016); Eliaz and Wu (2018); Crutzen, Flamand, and Sahuguet (2020); and Arbatskaya and Konishi (2023). In their settings, team performance is determined by a function aggregating efforts of all members.[10] While in ours, a team's performance is instead evaluated by the full path of battle outcomes, which differs from most studies in this stream

[8]Lemus and Marshall (2021) show that in online dynamic innovation procurement, performance feedback through a real-time public leaderboard on average improves competition outcomes.

[9]Klumpp, Konrad, and Solomon (2019) study sequential Blotto games with a majoritarian objective. They find that the history-independence result can extend to their setting with two individual contestants.

[10]Typical functions include maximum or minimum member performance, additively separable (possibly nonlinear) function, Cobb-Douglas function, and constant-elasticity-of-substitution production function.

of the literature. Moreover, we focus on the effort-maximizing prize design in team contests.

Our paper also speaks to the literature on biased contests and their optimal design, including Li and Yu (2012); Pastine and Pastine (2012); Franke, Kanzow, Leininger, and Schwartz (2013); Seel and Wasser (2014); Fu and Wu (2020), among others. These studies mainly concern the design of multiplicative biases and additive headstarts in Tullock contests. Our paper differs from the literature in two aspects. First, we consider a team contest setting with multiple pairwise battles. Second, we study the optimal prize allocation rule based on battle outcomes. We find that an additively biased prize allocation rule can be optimal when teams are sufficiently heterogeneous.

The rest of the paper is organized as follows. In Section 2, we set up the model. We study the optimal prize design in Section 3. Section 4 presents some major properties of the optimal design, illustrates several possible extensions, and discusses the main implications of our results. Section 5 concludes. The appendix collects some technical proofs.

2 The Model

Two teams, indexed by A and B, compete in a contest with N (odd or even) pairwise battles. Each team consists of N risk-neutral players, and each player on one team is matched to his opponent from the rival team. The matched players compete head-to-head on N disjoint battlefields. A player on team $i \in \{A, B\}$ is indexed by $i(t)$ if he is assigned to battle t, where $t \in \mathcal{N}$ and $\mathcal{N} \triangleq \{1, 2, ..., N\}$ denote the set of all battles. In each component battle t, two matched players simultaneously exert their efforts, $x_{A(t)}$ and $x_{B(t)}$. Player $i(t)$'s effort entry $x_{i(t)}$ incurs a constant marginal cost $c_{i(t)} > 0$, which is public information. We assume that component battles are carried out completely successively. As we consider a complete-information contest game, the solution concept is sub-game perfect equilibrium.

Given $x_{A(t)}$ and $x_{B(t)}$, player $i(t)$ wins the battle t with probability $p_{i(t)}(x_{A(t)}, x_{B(t)})$ such that $p_{A(t)}(x_{A(t)}, x_{B(t)}) + p_{B(t)}(x_{A(t)}, x_{B(t)}) = 1$. As in Fu, Lu, and Pan (2015), we assume that the winning probability is homogeneous of degree zero in players' efforts, allowing for various contest technologies.

Assumption 1 *$\forall x_{A(t)}, x_{B(t)} \geq 0$, $\theta > 0$, and $t \in \mathcal{N}$, $p_{i(t)}(\theta x_{A(t)}, \theta x_{B(t)}) = p_{i(t)}(x_{A(t)}, x_{B(t)})$.*

Apparently, in the generalized Tullock contest, the winning probability $p_{i(t)}(x_{i(t)}, x_{j(t)}) = x_{i(t)}^{r(t)} / \left(x_{i(t)}^{r(t)} + x_{j(t)}^{r(t)}\right)$ satisfies Assumption 1, where $r(t) > 0$ denotes the discriminatory power of battle t.

The contest organizer has a fixed budget, which is fully divisible and normalized as 1, to reward teams. In our analysis, we consider the team prize to be a *public good* that holds equal value for all players within the team. For simplicity of analysis, we assume that there is no private benefit for an individual player from winning his own battle. As a result, a player can only benefit from his team prize. This pure public-good setup at least serves as a good starting point of the analysis, and is a common scenario in many real-world team contests. For instance, in team sports, contestants are primarily motivated by their team's success. Similarly, in R&D race between research alliances, each research unit is mainly driven by their alliance's overall winning prospect.

The contest organizer aims to maximize the expected total effort of players from both teams by choosing a prize allocation rule and fully committing to it. The prize allocation rule can be contingent on the contest outcomes—i.e., the full path of battle outcomes. To better illustrate, we denote the set of winning battles of the concerned team i as subset $\mathcal{W}^i (\in 2^{\mathcal{N}})$, $\forall i \in \{A, B\}$. Apparently, if team i wins battles $\mathcal{W}^i$, team j $(\neq i)$ must win the remaining battles $\mathcal{W}^j = \mathcal{N} \backslash \mathcal{W}^i$, and there are 2^N possible outcomes in total. We use $v^i(\mathcal{W}^i)(\geq 0)$ to denote the prize allocated to team i.[11] Our prize allocation rule $(v^A(\cdot), v^B(\cdot))$ can be path-dependent since a team's prize is contingent on the full path of battle outcomes or the set of battles it wins, i.e., $v^i(\cdot) : 2^{\mathcal{N}} \to [0, 1]$. Throughout the paper, we restrict our attention to the prize allocations that satisfy nonnegativity, monotonicity, and budget balance conditions summarized in the following Assumption 2.

Assumption 2 *(i) Nonnegativity.* $v^i(\mathcal{W}^i) \geq 0$, $\forall \mathcal{W}^i$, $\forall i \in \{A, B\}$.

(ii) Monotonicity. $(\mathcal{W}^i)' \subseteq \mathcal{W}^i \Longrightarrow v^i((\mathcal{W}^i)') \leq v^i(\mathcal{W}^i)$, $\forall i \in \{A, B\}$.

(iii) Budget Balance. $v^i(\mathcal{W}^i) + v^j(\mathcal{W}^j) = 1$, $\forall \mathcal{W}^i$ *and* $\mathcal{W}^j = \mathcal{N} \backslash \mathcal{W}^i$, $i, j \in \{A, B\}$, $i \neq j$.

The nonnegativity condition requires that the prizes are nonnegative, and the mono-

[11] We assume that the prize to a team under each winning outcome must be nonnegative. See Moldovanu, Sela, and Shi (2012), Liu, Lu, Wang, and Zhang (2018), and Liu and Lu (2023) for analyses on negative prizes.

tonicity condition requires that additional victory is never detrimental. The budget balance condition requires that the prize budget is always exhausted, which plays a crucial role in our analysis. Due to the budget balance condition, team B's prize can be fully determined by team A's prize. Therefore, to search for the effort-maximizing prize allocation rule, it suffices to focus on team A's prize allocation rule $v^A(\cdot)$. More importantly, as will be revealed later, the budget balance condition implies that in our setting, all battles can be viewed as independent draws with fixed winning chances. This observation dramatically simplifies our analysis.

We say that a prize allocation rule $(v^A(\cdot), v^B(\cdot))$ is path-independent if and only if a team's prize is solely contingent on the number of battles each team wins, i.e., $v^i(\cdot) : \mathbf{N} \to [0, 1]$. Clearly, path-independent rules only make sense if all battles are symmetric. Let k^A be the number of winning battles of team A. Therefore, $v^A(\mathcal{W}^A)$ and $v^A(k^A)$ respectively denote the prize to team A for path-dependent and path-independent allocation rules. We will study how to design the effort-maximizing prize allocation rule subject to Assumption 2. The solution concept applied in this paper is subgame perfect Nash equilibrium.

Our model can be interpreted in alternative way. When the prize budget is indivisible, the prize to team i can be interpreted as team i's winning chance of the whole prize. Assumption 2(i) (nonnegativity) automatically holds as winning probabilities cannot be negative. Assumption 2(ii) (monotonicity) means that winning an additional battle does not decrease the team's winning chance. Assumption 2(iii) means that there must be a winner.

Designer's Objective Function

Fu, Lu, and Pan (2015) establish independence results in majoritarian multi-battle team contests. As a consequence, players' winning probabilities can be viewed as independent draws. This result extends to the settings with any number of battles and any feasible prize rule satisfying Assumption 2 whenever the contest success function is homogeneous of degree zero in the efforts. Relying on this generalized independence result, we are able to pin down the designer's objective function (total expected effort) as a linear function of prizes. The details are as follows.

Consider a path-dependent allocation rule $v^A(\cdot) : 2^{\mathcal{N}} \to [0, 1]$. The state of the contest

before battle t is summarized by a tuple $(\mathcal{N}_t, \mathcal{W}_t^A, \mathcal{W}_t^B)$, where $\mathcal{N}_t = \{1, 2, \cdots, t-1\}$ denotes the set of finished battles and $\mathcal{W}_t^i \subseteq \mathcal{N}_t$ the set of battles that team i wins. Since $\mathcal{W}_t^A \cup \mathcal{W}_t^B = \mathcal{N}_t$ and $\mathcal{W}_t^A \cap \mathcal{W}_t^B = \varnothing$, we simply use $(\mathcal{N}_t, \mathcal{W}_t^A)$ to represent the state. Denote $\mathbb{E}x_{i(t)}(\mathcal{W}_t^A)$ as player $i(t)$'s expected effort in battle t when the state is $(\mathcal{N}_t, \mathcal{W}_t^A)$. Therefore, the (ex-ante) expected total effort can be written as follows.

$$\mathbf{TE}(v^A) \triangleq \sum_{t \in \mathcal{N}} \sum_{\mathcal{W}_t^A \subseteq \mathcal{N}_t} \Pr(\mathcal{W}_t^A)[\mathbb{E}x_{A(t)}(\mathcal{W}_t^A) + \mathbb{E}x_{B(t)}(\mathcal{W}_t^A)], \tag{1}$$

where $\Pr(\mathcal{W}_t^A)$ is the probability that the state is $(\mathcal{N}_t, \mathcal{W}_t^A)$ before battle t.

Let $U_t^A(\mathcal{W}_{t+1}^A)$ denote the *expected prize* won by each player on team A *before* battle $t+1$ that has a history $\mathcal{W}_{t+1}^A$, and let $V_t(\mathcal{W}_t^A)$ denote player $A(t)$'s valuation of winning current battle t at state $(\mathcal{N}_t, \mathcal{W}_t^A)$. Since each player in a team contest turns up only once and bears no cost in future battles, the *prize spread* for battle t with $\mathcal{W}_t^A$ is merely $V_t(\mathcal{W}_t^A) = U_t^A(\mathcal{W}_t^A \cup \{t\}) - U_t^A(\mathcal{W}_t^A)$.

Consider a component battle t at state $(\mathcal{N}_t, \mathcal{W}_t^A)$, as in Observation 1 and Theorem 1 of Fu, Lu, and Pan (2015), it follows from the budget balance condition that two matched players have the same valuation of winning the current battle t, that is, $V_t(\mathcal{W}_t^A)$. In our context, we refer to this common valuation of winning as the *effective prize spread* of battle t. This result leads to the following property.

Property 1 *Given players' marginal effort costs and the contest technology, for all t, $\mathcal{W}_t^A$, there exist scalars α_t and $p_{A(t)}$ that depend solely on $c_{A(t)}, c_{B(t)}$ and the contest technology in battle t, such that at equilibrium (i) $\mathbb{E}x_{A(t)}(\mathcal{W}_t^A) + \mathbb{E}x_{B(t)}(\mathcal{W}_t^A) = \alpha_t V_t(\mathcal{W}_t^A)$; (ii) player $A(t)$ wins battle t with probability $p_{A(t)}$.*

Property 1(i) and (ii) talk about two different terms in Equation (1): $[\mathbb{E}x_{A(t)}(\mathcal{W}_t^A) + \mathbb{E}x_{B(t)}(\mathcal{W}_t^A)]$ and $\Pr(\mathcal{W}_t^A)$, respectively.[12] Due to Assumption 1, Property 1(i) means that the sum of players' expected efforts in each battle must be proportional to the prize spread. Moreover, the ratio α_t is dependent on t while invariant to the states. When the effective prize spread is positive, Property 1(ii) follows directly from the history independence result

[12]For the generalized Tullock contests, $p_{A(t)}$ and α_t are explicitly provided in Section 4.2.

by Fu, Lu, and Pan (2015). When the effective prize spread is zero, i.e., $V_t(\mathcal{W}_t^A) = 0$, we call such a battle $t|\mathcal{W}_t^A$ a trivial battle wherein players simply make zero effort. In the following lemma, we establish that it is without loss of generality to assume that player $A(t)$ wins with probability $p_{A(t)}$, even for trivial battles.

Lemma 1 (Trivial Battle) *If battle $t|\mathcal{W}_t^A$ is trivial, the expected total effort remains the same when the winning probabilities of the players in battle t are reset as $(p_{A(t)}, 1 - p_{A(t)})$.*

Proof. See the Appendix. ■

With Property 1(ii), ex-ante battle outcomes can be treated as independent lotteries, which inherits the merit of history independence in the literature. By direct calculation, the probability that $\mathcal{W}_t^A$ occurs is $\Pr(\mathcal{W}_t^A) = \prod_{j\in\mathcal{W}_t^A} p_{A(j)} \prod_{j\in\mathcal{N}_t\setminus\mathcal{W}_t^A}(1 - p_{A(j)})$. Therefore, the expected total effort in Equation (1) can be written as $\mathbf{TE}(v^A) = \sum_{t\in\mathcal{N}} \alpha_t \mathbf{PS}_t(v^A)$, where

$$\mathbf{PS}_t(v^A) = \sum_{\mathcal{W}_t^A\subseteq\mathcal{N}_t} \left\{ \prod_{j\in\mathcal{W}_t^A} p_{A(j)} \prod_{j\in\mathcal{N}_t\setminus\mathcal{W}_t^A} (1 - p_{A(j)}) \right\} V_t(\mathcal{W}_t^A)$$

denotes the (ex-ante) expected effective prize spread of battle t.

We then analyze the contest dynamics to pin down the analytical form of $V_t(\mathcal{W}_t^A)$ in $\mathbf{PS}_t(v^A)$. For this purpose, we track players' incentives by computing U_t^A backward. At the end of the contest, i.e., $t = N$, the continuation value coincides with the prize, which yields the boundary condition for U^A: $U_N^A(\mathcal{W}_{N+1}^A) = v^A(\mathcal{W}_{N+1}^A)$. Given an arbitrary battle t at state $(\mathcal{N}_t, \mathcal{W}_t^A)$, if player $A(t)$ wins, the contest reaches state $(\mathcal{N}_{t+1}, \mathcal{W}_t^A \cup \{t\})$ and the continuation value for team A's players becomes $U_t^A(\mathcal{W}_t^A \cup \{t\})$; if player $A(t)$ loses, the contest reaches state $(\mathcal{N}_{t+1}, \mathcal{W}_t^A)$ and the continuation value correspondingly becomes $U_t^A(\mathcal{W}_t^A)$. Since player $A(t)$ wins battle t with probability p_A regardless of the state (Property 1(ii)), we obtain the recursive definition for U^A: $U_{t-1}^A(\mathcal{W}_t^A) = p_{A(t)} U_t^A(\mathcal{W}_t^A \cup \{t\}) + \left(1 - p_{A(t)}\right) U_t^A(\mathcal{W}_t^A)$. Figure 1 illustrates the dynamics of the team contests.

Based on the boundary condition and recursive definition, we derive the analytical formulas of $U_{t-1}^A(\mathcal{W}_t^A)$ and $V_t(\mathcal{W}_t^A)$ in terms of prizes $\{v^A(\mathcal{W}^A)\}_{\mathcal{W}^A\in 2^{\mathcal{N}}}$ and further characterize $\mathbf{TE}(v^A)$ in terms of $\{v^A(\mathcal{W}^A)\}_{\mathcal{W}^A\in 2^{\mathcal{N}}}$. The result is summarized as follows.

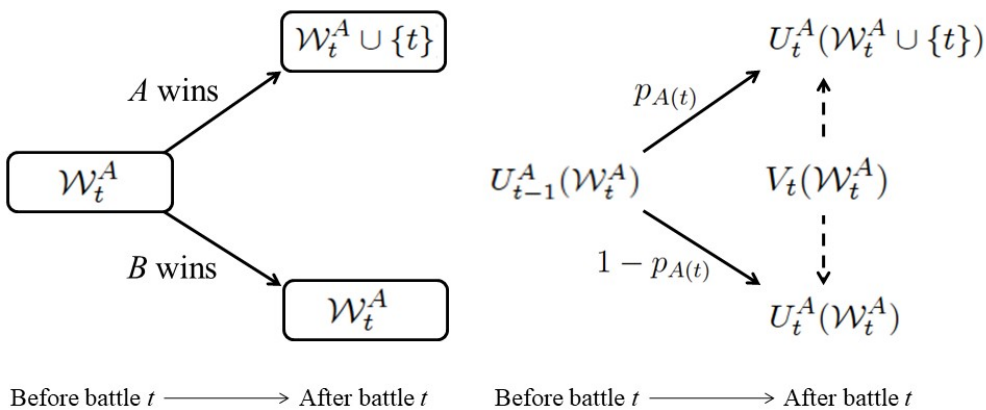

Figure 1: Dynamics of the Team Contest

Lemma 2 (Objective Function) *The expected total effort over all N battles, $\mathbf{TE}(v^A)$, is a linear function of $v^A(\mathcal{W}^A)$, $\forall \mathcal{W}^A \subset \mathcal{N}$. Specifically,*

$$\mathbf{TE}(v^A) = \sum_{t \in \mathcal{N}} \alpha_t \mathbf{PS}_t(v^A), \tag{2}$$

where

$$\mathbf{PS}_t(v^A) = \sum_{\mathcal{W}^A \subseteq \mathcal{N}} \left[(-1)^{\mathbf{1}(t \notin \mathcal{W}^A)} \left\{ \prod_{j \in \mathcal{W}^A, j \neq t} p_{A(j)} \prod_{j \notin \mathcal{W}^A, j \neq t} (1 - p_{A(j)}) \right\} v^A(\mathcal{W}^A) \right]. \tag{3}$$

Proof. See the Appendix. ■

Lemma 2 pins down the contest designer's objective as a linear function of $\{v^A(\mathcal{W}^A)\}_{\mathcal{W}^A \in 2^{\mathcal{N}}}$. Notice that $\mathbf{PS}_t(v^A) = \sum_{\mathcal{W}^A \subseteq \mathcal{N} \setminus \{t\}} \left\{ \prod_{j \in \mathcal{W}^A} p_{A(j)} \prod_{j \notin \mathcal{W}^A} (1 - p_{A(j)}) \left[v^A(\mathcal{W}^A \cup \{t\}) - v^A(\mathcal{W}^A) \right] \right\}$. In this alternative expression, $\prod_{j \in \mathcal{W}^A} p_{A(j)} \prod_{j \notin \mathcal{W}^A} (1 - p_{A(j)})$ represents team A's probability of winning $\mathcal{W}^A$ out of $\mathcal{N} \setminus \{t\}$, and $v^A(\mathcal{W}^A \cup \{t\}) - v^A(\mathcal{W}^A)$ then denotes the effective prize spread of battle t. Therefore, $\mathbf{PS}_t(v^A)$ is the (ex-ante) expected effective prize spread of battle t given that the outcomes of all battles except t were drawn independently.

3 Optimal Prize Design

Based on the designer's objective function of total effort maximization, we first provide a fundamental property of the optimal prize rule in Section 3.1, which rules out the possibility of split prizes between teams. Then we proceed to characterize the optimal prize design in Section 3.2. In Section 3.3, we further show that our analysis can be extended to solving a problem of total weighted effort maximization, in which the organizer values the battle efforts differently. Lastly, we examine an important special case of homogeneous battles in Section 3.4.

3.1 Simplifying the Problem

To formally characterize the optimal rule, we first demonstrate that at optimum each prize must be either 0 or 1, i.e., a team is awarded either the whole prize or nothing. This rules out the possibility of partial or split prizes. Based on this observation, we can therefore define the winner and loser of the whole contest: A team is called the winner if and only if it acquires the entire prize.

Lemma 3 (Win or Lose) *With full-fledged heterogeneity, there must exist an optimal prize allocation rule $v^A(\cdot)$ such that $v^A(\mathcal{W}^A) \in \{0, 1\}$.*

Proof. See the Appendix. ■

A sketch proof is provided below, which consists of two steps. First, we argue that the set of all v^A functions, denoted by $\mathcal{V}^A$, is convex and $\mathbf{TE}(v^A)$ is linear in v^A within $\mathcal{V}^A$. The convexity of $\mathcal{V}^A$ follows directly from three conditions in Assumption 2, which are all linear. The linearity of $\mathbf{TE}(v^A)$ in v^A is a consequence of history independence result as well as Equation (3). By the fundamental theorem of linear programming, the maximum effort level can always be attained at the vertices of $\mathcal{V}^A$.[13]

Second, we show that every v^A that belongs to the vertices of $\mathcal{V}^A$ must satisfy that $v^A(\mathcal{W}^A) \in \{0, 1\}$. Since the number of vertices is finite (no larger than 2^{2^N}), the optimal prize allocation rule can always be attained by vertices of $\mathcal{V}^A$. In other words, an allocation

[13]Fundamental theorem of linear programming says that a linear objective function f defined over a polygonal convex set attains a maximum (or minimum) value at a corner point of the set.

with split prizes can be decomposed into a convex combination of vertex rules that exclude these prizes.

Lemma 3 facilitates our search for the optimal prize rules tremendously since it reduces the number of candidates from infinite to finite, which makes enumeration possible. Based on the result, we can therefore define the winner and loser of the team contest: A team is called the winner if it acquires the entire prize.

3.2 Optimal Design

Due to full-fledged heterogeneity, it is not straightforward to tell which team is stronger as a whole, since a team may contain both weaker and stronger players, relative to their opponents in the rival team. Nevertheless, we propose a measurement of team strength that aggregates all relevant information to determine which team is stronger and by how much. This step plays a key role in characterizing and interpreting the optimal rule.

We first introduce the definition of player strength, which is the building block to evaluate a team's strength.

Definition 1 (Player Strength) *Player $i(t)$'s strength is defined as $s_{i(t)} \triangleq \frac{\alpha_t}{1-p_{i(t)}}, \forall i, t$.*

Recall that α_t denotes the ratio of the total effort in battle t to the effective prize spread of battle t (see Property 1(i)). Intuitively, α_t should increase with the players' total strength in battle t, and decreases with their degree of asymmetry. As a result, players' total strength should increase with α_t and their degree of asymmetry. Since $p_{A(t)} + p_{B(t)} = 1$, then $\frac{1}{p_{A(t)}p_{B(t)}}$ naturally measures the degree of asymmetry across the two players. It follows that

$$s_t = \frac{\alpha_t}{p_{A(t)}p_{B(t)}}$$

becomes a natural measure for the total strengths of two players in battle t. We can split this total strength between the two players according to their winning probabilities. Then, we have the player $i(t)$'s strength $s_{i(t)} = p_{i(t)}s_t = \frac{\alpha_t}{1-p_{i(t)}}$.

By summing up all players' strengths in a team, we can further define the *team strength* for each team.

Definition 2 (Team Strength) *Team i's team strength is defined as $S_i \triangleq \sum_{t\in\mathcal{N}} s_{i(t)}, \forall i$.*

Note that S_i is solely determined by the model primitives $\left\{c_{A(t)}, c_{B(t)}, r(t)\right\}_{t=1}^{N}$. Relying on this definition, the team with higher team strength is the stronger team. Nevertheless, $S_i > S_j$ does not imply that $p_{i(t)} > p_{j(t)}$ holds uniformly across all battles. The definition of team strength thus converts players' strengths within a team into a single-dimensional measure. Without loss of generality, we assume in the subsequent analysis that team A is the stronger team, i.e., $S_A \geq S_B$.

We next introduce a class of winning rules called majority-score rule with a headstart: each battle is assigned a team-invariant score, a headstart score is assigned to the weaker team as favoritism, the team collecting higher total scores from its winning battles wins the entire prize.

Definition 3 (Majority-score Rule with a Headstart) *In a multi-battle team contest, each battle is assigned a score w_t. Let $w^A(\mathcal{W}^A) = \sum_{t\in\mathcal{W}^A} w_t$ and $w^B(\mathcal{W}^B) = \sum_{t\in\mathcal{W}^B} w_t$ denote the sum of scores won by teams A and B, respectively. $H \geq 0$ denotes the headstart score allocated to the weaker team B. Team A (the stronger team) collects the whole prize budget if $w^A(\mathcal{W}^A) > w^B(\mathcal{W}^B) + H$; and team B collects the whole prize budget if $w^A(\mathcal{W}^A) < w^B(\mathcal{W}^B) + H$; when $w^A(\mathcal{W}^A) = w^B(\mathcal{W}^B) + H$, the tie can be broken arbitrarily.*

This class of winning rule is commonly observed in practice. If we take the US presidential election as a multi-battle contest, then the winner of a state collects scores equal to the electoral votes, and the candidate or party with a higher total score wins the election. In addition, headstart is well documented in the literature, and it is often included in the design.[14] In score contests, the presence of a headstart in the form of initial scores is quite common. One example of such headstarts is the practice of partisan gerrymandering.[15]

In the following main theorem of the paper, we show that the optimal design must fall in the class of the majority-score rules with a headstart; moreover, we fully pin down the optimal battle scores and the initial headstart score.

[14] Heating up an unbalanced competition in asymmetric contests through headstart is widely studied in the literature. See Li and Yu (2012); Pastine and Pastine (2012); and Seel and Wasser (2014), among others.

[15] Gerrymandering helps secure wins in some battles for the ruling party, which can be regarded as providing it a headstart.

Theorem 1 (Optimality of Majority-score Rule with a Headstart) *With full-fledged heterogeneity, the optimal allocation rule is a* ***majority-score rule with a headstart****, in which* $w_t = s_t, \forall t$ *and* $H = S_A - S_B$.

Proof. We provide a sketch of proof here. Details are relegated to the Appendix.

For every prize allocation rule satisfying $v^A(\mathcal{W}^A) \in \{0, 1\}$ (see Lemma 3), we can always find a *minimum winning outcome* $\underline{\mathcal{W}}^A$ such that $v^A(\underline{\mathcal{W}}^A) = 1$ and $v^A(\mathcal{W}^A) = 0$ for all $\mathcal{W}^A \subsetneqq \underline{\mathcal{W}}^A$. Similarly, we can define the *maximum losing outcome* $\overline{\mathcal{W}}^A$ such that $v^A(\overline{\mathcal{W}}^A) = 0$ and $v^A(\mathcal{W}^A) = 1$ for all $\mathcal{W}^A \supsetneqq \overline{\mathcal{W}}^A$.

We then show that changing from $v^A(\underline{\mathcal{W}}^A) = 1$ to $v^A(\underline{\mathcal{W}}^A) = 0$, which does not violate any constraint, would increase the total effort level if $w^A(\underline{\mathcal{W}}^A) < w^B(\underline{\mathcal{W}}^A) + H$. Hence, the optimal design must have $w^A(\underline{\mathcal{W}}^A) \geq w^B(\underline{\mathcal{W}}^A) + H$ for all minimum winning outcomes. This suffices to show that $w^A(\mathcal{W}^A) \geq w^B(\mathcal{W}^A) + H$ holds for all $\mathcal{W}^A$ such that $v^A(\mathcal{W}^A) = 1$. Similarly, $w^A(\mathcal{W}^A) \leq w^B(\mathcal{W}^A) + H$ holds for all $\mathcal{W}^A$ such that $v^A(\underline{\mathcal{W}}^A) = 0$. ■

Equivalently, the optimal prizes are $v^A(\mathcal{W}^A) = \begin{cases} 1, & \text{if } w^A(\mathcal{W}^A) > S_A, \\ 0, & \text{if } w^A(\mathcal{W}^A) < S_A, \\ 0 \text{ or } 1, & \text{if } w^A(\mathcal{W}^A) = S_A. \end{cases}$ and $v^B(\mathcal{W}^B) = 1 - v^A(\mathcal{W}^A)$, since $S_A + S_B = \sum_{t \in \mathcal{N}} w_t$ by definition. In other words, the team strengths S_A and S_B can be viewed as the respective *winning thresholds* for two teams. Since $s_{i(t)} = p_{i(t)} w_t$ (by construction) is the expected battle score obtained by player $i(t)$, $S_i \triangleq \sum_{t \in \mathcal{N}} s_{i(t)}$ is simply the *expected team score* obtained by all members on team i.

Remark 1 $S_i,\ i \in \{A, B\}$ *equals team* i*'s* ***winning threshold score****, which in turn coincides with its* ***expected team score****.*

Remark 1 provides an interesting alternative interpretation of the optimal prize allocation rule: the optimal prize rule actually rewards the entire prize to the team that outperforms its expected score, i.e. its expected performance.

The optimal rule incentivizes effort supply through two instruments: the score of each battle w_t and the winning threshold for each team S_i (in terms of unadjusted total scores). To see how the optimal rule in Theorem 1 reacts to the heterogeneity within a battle through the two instruments, consider an unbalanced battle t in which player $i(t)$ is stronger than his

opponent $j(t)$, i.e., $c_{i(t)} < c_{j(t)}$ or $p_{i(t)} > p_{j(t)}$. For simplicity, in the following discussion, we focus on battle structure changes (costs and technology) that only affect α_t or $p_{i(t)}$.[16] When α_t increases, the effort becomes more effective in determining the outcome, and the optimal prize rule raises score w_t in battle t because a higher score should be set to provide a higher incentive in such a battle. In addition, w_t should always increase whenever battle t becomes more unbalanced. In particular, when $p_{i(t)}$ increases, the *degree of imbalance* measured by $1/(p_{A(t)}p_{B(t)})$ also increases and the optimal prize rule assigns a greater score w_t to battle t. Moreover, the higher $p_{i(t)}$ is, the higher player strength $s_{i(t)}$ and also the higher winning threshold S_i would be. In the meanwhile, since $p_{j(t)} = 1 - p_{i(t)}$, the weaker player's strength $s_{j(t)}$ and the winning threshold S_j both decrease. Our result reveals that headstart is set to counterbalance the asymmetry between two teams: if team i is the stronger team (i.e., team j is the weaker team), headstart to the weaker team j should increase with $p_{i(t)}$; if team i is the weaker team and headstart to the weaker team i should instead decrease with $p_{i(t)}$.

3.3 Total Weighted Effort Maximization

In contests involving teams, battles can occur in various dimensions, areas, or activities. The organizer may assign different values to the efforts put forth in these battles. For instance, different stages of a multi-stage innovation tournament between research alliances could be rated differently by organizers. Similarly, sequential sports matches might receive varying levels of attention from audiences, leading organizers to evaluate efforts along the matches differently. Additionally, an organizer could place different weights on the efforts of different teams in the same battle. To account for these considerations, this subsection expands Theorem 1 by introducing a more comprehensive framework that accommodates varying weights assigned to different players' efforts in contests.

Consider the situation in which the contest organizer wishes to maximize the total weighted efforts across battles, i.e., $\sum_{t\in\mathcal{N}}(d_{A(t)}x_{A(t)} + d_{B(t)}x_{B(t)})$, where $d_{i(t)} > 0$ denotes the *effort weight* of player $i(t)$. If $d_{A(t)} = d_{B(t)}$, we denote this common weight by d_t, which

[16] For example, given the contest technology, we can proportionally reduce $c_{A(t)}$ and $c_{B(t)}$ to increase α_t while fixing $p_{A(t)}$. Meanwhile, appropriately decreasing $c_{i(t)}$ and increasing $c_{j(t)}$ can increase $p_{i(t)}$ without changing α_t. Please refer to Section 4.2 for more details.

is the effort weight of battle t. As pointed out by Fu, Lu, and Pan (2015), Assumption 1 implies that equilibrium strategies are homogeneous of degree one in the effective prize spread. Let $\mathbb{E}x_{A(t)}(\mathcal{W}_t^A) = \alpha_{A(t)}V_t(\mathcal{W}_t^A)$ and $\mathbb{E}x_{B(t)}(\mathcal{W}_t^A) = \alpha_{B(t)}V_t(\mathcal{W}_t^A)$. Hence, $\alpha_t = \alpha_{A(t)} + \alpha_{B(t)}$.

To investigate the optimal design with effort weights, we first formulate the objective function faced by the contest organizer as $\sum_{t\in\mathcal{N}} z_t \mathbf{PS}_t(v^A)$, where $z_t \triangleq d_{A(t)}\alpha_{A(t)} + d_{B(t)}\alpha_{B(t)}$. By replacing α_t by z_t in Equation (2), we obtain the weighted total effort. In particular, when the effort exerted by two players in the same battle is evaluated equally (i.e., $z_t = d_t\alpha_t$), the weighted total effort can be obtained by multiplying α_t by d_t in Equation (2). In this case, the effort weight plays a similar role as α_t (i.e. the ratio of total effort to effective prize spread) in determining the optimal design.

With full heterogeneity in effort evaluation, we simply replace α_t with z_t (i.e., $d_t\alpha_t$ or $d_{A(t)}\alpha_{A(t)} + d_{B(t)}\alpha_{B(t)}$) and apply the aforementioned procedure to derive the optimal design in Theorem 1. More precisely, the strength of $i(t)$ is now given by $s_{i(t)}^{\mathrm{W}} \triangleq \frac{z_t}{1-p_{i(t)}}$, and the team strength is still defined as $S_i^{\mathrm{W}} = \sum_{t\in\mathcal{N}} s_{i(t)}^{\mathrm{W}}$. The score assigned to battle t is now $w_t^{\mathrm{W}} \triangleq \frac{z_t}{p_{A(t)}p_{B(t)}}$, which increases with the effort weight. By inserting these updates into w^A, w^B, and H, Theorem 1 can easily accommodate effort heterogeneity across battles. Specifically, $H = \sum_{t\in\mathcal{N}}[s_{A(t)}^{\mathrm{W}} - s_{B(t)}^{\mathrm{W}}] = \sum_{t\in\mathcal{N}} \frac{(p_{A(t)}-p_{B(t)})z_t}{p_{A(t)}p_{B(t)}}$; thus, as $d_{A(t)}$ or $d_{B(t)}$ grows, H adjusts to give an additional advantage to the team that includes the weaker player in battle t. Relying on the analysis of Section 3.2, we restate our result in the following proposition.

Proposition 1 *With full-fledged heterogeneity, consider a total weighted effort maximization problem with weight $d_{i(t)} > 0$ for player $i(t)$, the optimal design can be obtained by simply replacing α_t with z_t in the Section 3.2. In particular, the score assigned to battle t increases with its own weights ($d_{A(t)}, d_{B(t)}$ or d_t) and does not depend on other battles' weights.*

We would like to emphasize that the generalization described above can be used to optimize the sum of the higher effort in each battle, provided that equilibrium is in pure strategies. This formalization well captures R&D scenarios where the designer aims to maximize the higher effort in each battle. To see that, we simply let $(d_{A(t)}, d_{B(t)}) = (1, 0)$ when $\alpha_t^A \geq \alpha_t^B$ and $(d_{A(t)}, d_{B(t)}) = (0, 1)$ when $\alpha_t^A < \alpha_t^B$.[17]

[17]Majority-score rule with a headstart is optimal even for mixed strategies. Equilibrium strategies are ho-

3.4 Homogeneous Battles

In this subsection, we consider a prominent special case where the contest technology is uniform across all battles and players on each team are homogeneous, while the two competing teams can be asymmetric in terms of players' marginal effort costs. Without loss of generality, we assume that team A is stronger than team B—i.e., team-A players have a lower marginal cost of effort denoted by $c_A \in (0, 1]$—while we normalize team-B players' marginal effort cost as $c_B = 1$. This setting allows us to concentrate on the asymmetry between rival teams.

With homogeneous battles, the number of winning battles is a sufficient statistic to determine the winning team. Formally, we can show that the expected total effort resulting from any path-dependent prize allocation rule can be duplicated by a path-independent allocation rule.[18] Therefore, providing a headstart in the form of initial wins is a simple way to balance two counterparties, as is seen in Asian handicap betting.[19] Such kind of majority rule is widely used in multi-battle contests such as tennis, volleyball, and snooker. We propose this winning rule in the following definition.

Definition 4 (Majority Rule with a Headstart) *In a majority rule with a headstart, team A will be allocated the entire prize if it wins at least $K_S(> \frac{N}{2})$ battles; otherwise, the entire prize is allocated to team B. Equivalently, the weaker team is given a headstart in the form of $2K_S - N - 1$ initial wins, and the entire prize is awarded to the team with the higher number of wins.*

The above rule simply gives the weaker team an additive headstart with size $2K_S - N - 1$.[20] Intuitively, the contest organizer levels the playing field by offering a headstart to the weaker.

mogeneous of degree one in the prize spread, meaning that a change in the prize spread scales the equilibrium strategies by the same factor. As a result, the higher effort in each battle is also scaled by this factor.

[18] According to Equation (3), one can easily derive that the coefficient of $v^A(\mathcal{W}^A)$ in the objective function is determined only by the number of winning battles, rather than the full path of battle wins.

[19] Asian handicap betting is a form of betting on sports in which the stronger team is handicapped so that it must win by more sets or matches in a multi-battle contest to win a bet. It uses a handicap system to give one team an advantage over the other, thus making the odds more even.

[20] If team A wins at least K_S battles and team B wins at most $N - K_S$ battles, then team B gains no more than $K_S - 1$ wins after counting in the headstart; if team A wins at most $K_S - 1$ battles and team B wins at least $N - K_S + 1$ battles, then team B gains at least K_S scores after counting in the additive headstart. In either case, tie never occurs.

In particular, a majority rule with a headstart degenerates into the conventional majority rule if no headstart is given and battles are of odd length.

When battles are homogeneous, we simply use α and p_A, where $\alpha_t = \alpha$ and $p_{A(t)} = p_A$ for all t. Using the notations, the score of each battle equals $w_t = \frac{\alpha}{p_A(1-p_A)}$, which is same across battles, and the threshold for team A is $S_A = \frac{N\alpha}{1-p_A}$. By applying Theorem 1, $w^A(\mathcal{W}^A) > (\leq) S_A$ if and only if $\frac{\alpha}{p_A(1-p_A)}|\mathcal{W}^A| > (\leq)\frac{N\alpha}{1-p_A}$, i.e., $|\mathcal{W}^A| > (\leq) p_A N$. Then we can present the optimal prize allocation rule in the following proposition.

Proposition 2 (Optimality of Majority Rule with a Headstart) *With homogeneous battles, the optimal allocation rule is a* ***majority rule with a headstart****, in which the headstart (in terms of initial wins) is* $\breve{H} = 2K_s - N - 1$*, where* $K_S = \lfloor p_A N \rfloor + 1$*.*

Proposition 2 shows the optimality of the path-independent rule and says that the K_S-th (resp. $(N - K_S + 1)$-th) win is critical for team A (resp. team B) in terms of the number of actual winning battles. Since $K_S \geq N - K_S + 1$, the majority rule with a headstart favors the weaker team, which in turn stimulates the stronger team.

4 Properties, Extensions, and Implications

In this section, we first provide extra properties of our optimal design in Section 4.1, Section 4.2, and Section 4.3. Following that, Section 4.4 provides three extensions to the contest designer's optimization problem. Finally, we discuss our main implications in Section 4.5.

4.1 Implementation through Elo Rating

We have established that the optimal design is a majority-score rule with a headstart. In the following, we implement the optimal design using Elo rating. Elo rating is widely adopted by many renowned sports associations, including FIFA and FIDE (World Chess Federation), to rate participants in bilateral games. Elo rating points are relative measures (rather than absolute measures) of the players' strengths within the rating pool.

The key procedure of an Elo rating system is the rating update process. To see how it works, consider player x after match t has been played, the change in his rating points Δ_x

is given by

$$\Delta_x = \mathbb{W}_t(R_x - \mathbb{P}_x), \tag{4}$$

where $\mathbb{W}_t$ is the weight assigned to battle t to measure the importance of the battle, $R_x \in \{0, 1\}$ is the outcome of the match from the perspective of player x (1 means a victory, and 0 means a loss), and $\mathbb{P}_x$ is the expected winning probability of player x.

An Elo rating system has three features: zero-sum, favoritism, and martingale. First, rating points are transferred from the loser to the winner. Second, a weaker player collects more points than a stronger player does through a victory. Third, the expected total change in points of the two players must be zero for each match.[21]

We next show how to implement our optimal design through Elo rating. Consider a battle t in the team contest, with $w_t, p_{A(t)}$ and $p_{B(t)}$ defined as before in Section 3. To apply the rating update formula Equation (4), let $\mathbb{W}_t = w_t, \mathbb{P}_{A(t)} = p_{A(t)}$ and $\mathbb{P}_{B(t)} = p_{B(t)}$. If player $A(t)$ wins battle t, the changes in the two players' rating points are

$$(\Delta_{A(t)}, \Delta_{B(t)}) = \big(w_t(1 - p_{A(t)}), w_t(0 - p_{B(t)})\big) = (s_{B(t)}, -s_{B(t)}) = (w_t, 0) - (s_{A(t)}, s_{B(t)}).$$

Otherwise, player $B(t)$ wins the battle, and the changes in the points are

$$(\Delta_{A(t)}, \Delta_{B(t)}) = \big(w_t(0 - p_{A(t)}), w_t(1 - p_{B(t)})\big) = (-s_{A(t)}, s_{A(t)}) = (0, w_t) - (s_{A(t)}, s_{B(t)}).$$

Apparently, the changes in points $\Delta_{A(t)}$ and $\Delta_{B(t)}$ are the additional scores, relative to the expected battle scores $s_{A(t)}$ and $s_{B(t)}$, earned by players $A(t)$ and $B(t)$, respectively. Section 3.2 establishes that the optimal rule rewards the entire prize to the team collecting a sufficient amount of scores that exceed its expected team score.[22] Equivalently, the optimal rule grants the entire prize purse to the team if its change in rating points is positive in the Elo system, as formulated in Theorem 2. Recall $R_{i(t)}$ denotes player $i(t)$'s winning outcome.

[21] Consider, for example, two players x and y, competing in a single match with importance $\mathbb{W}_t = 32$. The expected winning probabilities of x and y are 80% and 20%, respectively. If x wins the match, he will obtain $\Delta_x = 32 \times (1 - 0.8) = 6.4$ rating points and y will receive $\Delta_y = 32 \times (0 - 0.2) = -6.4$. Otherwise, y wins the match and extracts $32 \times (1 - 0.2) = 25.6$ points from x. The expected change in points for player x is $6.4 \times 0.8 + (-25.6) \times 0.2 = 0$, and for player y is $(-6.4) \times 0.8 + 25.6 \times 0.2 = 0$.

[22] Please refer to the alternative interpretation of the optimal prize allocation rule following Remark 1.

Theorem 2 (Implementation through Elo Rating System) *With full-fledged heterogeneity, we define the change in player $i(t)$'s Elo rating points as $\Delta_{i(t)} = w_t(R_{i(t)} - p_{i(t)})$ and the change in team i's Elo rating points as $\Delta_i \triangleq \sum_{t\in\mathcal{N}} \Delta_{i(t)}$. The optimal prize allocation rule rewards the entire prize to team i if and only if $\Delta_i > 0$.*

The Elo scoring system illustrates an alternative way to put our optimal majority-score rule with a headstart into work in competitive environments. It indicates that our optimal prize rule could be easily implemented in reality.

4.2 Comparative Statics

To study how battle characteristics, including contest technology and cost parameters, affect optimal design, we consider the family of generalized Tullock contests. The analytical formulas for α_t and $p_{A(t)}$ are derived from Lemma 1 in Feng and Lu (2018). Let $\widehat{r}(z) \in (1,2)$ represent the unique solution to $r = 1 + z^r$ with $z \in (0,1]$. If $c_{A(t)} \geq c_{B(t)}$,

$$p_{A(t)} = \begin{cases} c_{B(t)}^{r(t)} / \left(c_{A(t)}^{r(t)} + c_{B(t)}^{r(t)}\right), & \text{if} \quad r(t) \leq \widehat{r}\left(c_{B(t)}/c_{A(t)}\right), \\ (r(t)-1)^{1-1/r(t)} c_{B(t)} / \left(r(t) c_{A(t)}\right), & \text{if} \quad r(t) \in (\widehat{r}\left(c_{B(t)}/c_{A(t)}\right), 2], \\ c_{B(t)} / \left(2c_{A(t)}\right), & \text{if} \quad r(t) > 2. \end{cases}$$

and

$$\alpha_t = \begin{cases} r(t) c_{A(t)}^{r(t)-1} c_{B(t)}^{r(t)-1} \left(c_{A(t)} + c_{B(t)}\right) \left(c_{A(t)}^{r(t)} + c_{B(t)}^{r(t)}\right)^{-2}, & \text{if} \quad r(t) \leq \widehat{r}\left(c_{B(t)}/c_{A(t)}\right), \\ (r(t)-1)^{1-1/r(t)} \left(c_{A(t)} + c_{B(t)}\right) / \left(r(t) c_{A(t)}^2\right), & \text{if} \quad r(t) \in (\widehat{r}\left(c_{B(t)}/c_{A(t)}\right), 2], \\ \left(c_{A(t)} + c_{B(t)}\right) / \left(2c_{A(t)}^2\right), & \text{if} \quad r(t) > 2. \end{cases}$$

The case of $c_{A(t)} < c_{B(t)}$ is analogous to the case of $c_{A(t)} \geq c_{B(t)}$.

4.2.1 Full-fledged Heterogeneity

In this part, we allow full heterogeneity across players and battles to investigate how battle characteristics impact the optimal design. Note that $s_{A(t)}$, $s_{B(t)}$ and w_t only vary with t through the parameters $c_{A(t)}$, $c_{B(t)}$, $r(t)$. While, the headstart score $H = \sum_{t\in\mathcal{N}} \Delta s_t$

with $\Delta s_t \triangleq s_{A(t)} - s_{B(t)}$, depends on $s_{A(t)}$ and $s_{B(t)}$ in each battle t. As a consequence, the structural parameters of battle t affect the headstart score H through Δs_t.

We first summarize the impact of marginal costs $c_{A(t)},\ c_{B(t)}$ in Proposition 3 when $r(t) = 1$, which means that battle t is a standard lottery contest.

Proposition 3 *When $r(t) = 1$, the battle score w_t decreases with $c_{A(t)}$ and $c_{B(t)}$; the headstart score H (to team B) decreases with $c_{A(t)}$ but increases with $c_{B(t)}$.*

Proof. See the Appendix. ■

According to Proposition 3, when a player in a battle becomes stronger as his marginal effort cost decreases, the importance of the concerned battle to the entire contest, measured by w_t, must increase. This echoes our early insight that the battle score should increase with battle productivity (α_t). Moreover, if a player on team A (the stronger team) grows stronger or a player on team B (the weaker team) becomes weaker, the headstart score (to team B) should increase to further favor team B, which conforms with the favoritism argument.

Analogously, we establish the results on the impact of discriminatory power $r(t)$, which are summarized in Proposition 4.

Proposition 4 *When $r(t) \leq 2$, the battle score w_t increases with $r(t)$; the headstart score H (to the weaker team B) increases with $r(t)$ if $c_{A(t)} < c_{B(t)}$ and decreases with $r(t)$ if $c_{A(t)} > c_{B(t)}$. When $r(t) > 2$, both the battle score and the headstart score remain unchanged.*

Proof. See the Appendix. ■

When a component battle is relatively noisy (i.e., lower $r(t)$), an underdog player in the concerned battle becomes weaker as the discriminatory power $r(t)$ increases. By the favoritism argument, the headstart score should adjust to further favor the team containing this underdog player. In contrast, when the contest is sufficiently discriminatory, both the battle score and the headstart score are independent of the discriminatory power, since the equilibrium effort and winning probabilities remain constant in this case.

4.2.2 Homogeneous Battles

With homogeneous battles, players within each team are homogeneous and all battles share the same discriminatory power $r \in (0, +\infty]$. All players in team $i \in \{A, B\}$ have

marginal effort cost c_i. In this case, player $A(t)$ wins with a battle-irrelevant probability p_A:

$$p_A = \begin{cases} 1/\left(1+c_A^r\right), & \text{if} \quad r \leq \widehat{r}(c_A), \\ 1-(r-1)^{1-1/r}\, c_A/r, & \text{if} \quad r \in (\widehat{r}(c_A), 2], \\ 1-c_A/2, & \text{if} \quad r > 2. \end{cases} \tag{5}$$

Section 3.4 shows that the optimal allocation rule intensifies the competition by compensating the underdog team and thus disciplining the favorite team. Specifically, the contest designer mitigates the asymmetry between the teams by awarding the weaker team a headstart to heat up the competition. The following question thus arises: How does the level of headstart respond as the two teams become more uneven?

Recall that Proposition 2 states that the minimum winning requirement for team A is $K_S = \lfloor p_A N \rfloor + 1$. Clearly, $\lfloor p_A N \rfloor$ (weakly) increases with winning probability p_A, so does K_S.

Proposition 5 *In the optimal rule (i.e., majority rule with a headstart), the minimum number of winning battles K_S for the stronger team A to win the contest (weakly) increases with p_A. In terms of model primitives, K_S (weakly) increases with r but (weakly) decreases with c_A.*

Proposition 5 demonstrates that when p_A increases, a higher K_S should be set to induce more effort. Recall that the contest organizer uses the majority rule with a headstart with K_S to favor the weaker team B, in order to balance the contest between two asymmetric teams. When the disparity in capabilities of the two teams (measured by p_A) increases, more favoritism should be offered to the weaker team to balance the contest.

Recall that p_A is determined by the marginal effort costs of the two teams and the discriminatory power of Tullock contests, whenever a battle is not trivial (see Equation (5)). Note that p_A decreases with c_A and (weakly) increases with r. When the two teams become more asymmetric (i.e., a lower c_A) or the contest becomes discriminatory (i.e., a higher r), Proposition 5 implies that a higher K_S should be set to induce more effort.

4.3 Majority Rule and Unanimous Rule

4.3.1 Full-fledged Heterogeneity

The best-of-N rule (the simple majority rule), is widely adopted in practice due to simplicity and fairness. Proposition 6 will provide the condition for a majority rule to be optimal. This analysis offers a theoretical justification for this popular contest rule.

Proposition 6 *With full-fledged heterogeneity, when the number of battles is odd, the simple majority rule is optimal if and only if scores assigned to all battles are sufficiently close (i.e.,* $\sum_{t=1}^{(N+1)/2} w_{(t)} \geq \sum_{t=(N+3)/2}^{N} w_{(t)}$*), and the difference in team strengths is sufficiently low (i.e.,* $H \leq \sum_{t=1}^{(N+1)/2} w_{(t)} - \sum_{t=(N+3)/2}^{N} w_{(t)}$*), where* $w_{(t)}$ *denotes the* t*-th minimum score among all battles.*

The condition $\sum_{t=1}^{(N+1)/2} w_{(t)} \geq \sum_{t=(N+3)/2}^{N} w_{(t)}$ is equivalent to $\frac{N+1}{N}\bar{w}_{\mathrm{L}} \geq \bar{w} \geq \frac{N-1}{N}\bar{w}_{\mathrm{H}}$, where $\bar{w}_{\mathrm{L}} = \frac{2}{N+1}\sum_{t=1}^{(N+1)/2} w_{(t)}$ is the average score in those less weighted battles, $\bar{w}_{\mathrm{H}} = \frac{2}{N-1}\sum_{t=(N+3)/2}^{N} w_{(t)}$ is the average score in those heavier weighted battles, and $\bar{w} = \frac{1}{N}\sum_{t=1}^{N} w_{(t)}$ is the average score in all battles. This condition requires that the scores assigned to battles are sufficiently close.

According to Proposition 6, the optimality of the simple majority rule rests on whether scores are relatively symmetric across battles. Note that a lopsided contest and a balanced contest can share very similar battle scores, and the simple majority rule could be optimal even when the winning probabilities vary dramatically across matches.

Example. Consider a 3-battle team contest and each battle is modeled as a lottery contest (i.e., Tullock contest with discriminatory power $r = 1$). Marginal costs and other parameters are summarized in Table 1. We can validate the conditions in Proposition 6 and pin down the optimal prize allocation rule is the simple majority rule.[23] Note that in this example team A dominates the first battle, team B prevails in the second, and both teams are evenly matched in the third battle.

We next turn to the unanimous rule that requires the stronger team (team A) to win all battles to gain the entire prize. We borrow the word "unanimous" from the voting literature

[23] The first condition is satisfied since $1 + 1.25 > 1.25$, and the second condition is fulfilled because the two teams have the same team strengths equaling 1.75.

Table 1: The Simple Majority Rule is Optimal

	$c_{A(t)}$	$c_{B(t)}$	$p_{A(t)}$	$p_{B(t)}$	α_t	$s_{A(t)}$	$s_{B(t)}$	w_t
Battle 1	1	4	0.8	0.2	0.2	1	0.25	1.25
Battle 2	4	1	0.2	0.8	0.2	0.25	1	1.25
Battle 3	2	2	0.5	0.5	0.25	0.5	0.5	1

to define the prize allocation rule that demands all battles to reach a "consensus" on their outcomes. In our context, the unanimous rule means that a stronger team receives nothing if it losses an arbitrary battle. Alternatively, the weaker team (team B) only needs to win one battle to win the whole prize. We have the following result on the sub-optimality of unanimous rule.

Proposition 7 *With full-fledged heterogeneity, the unanimous rule can never be optimal if the weaker team dominates strictly more than one battle.*

Proof. See the Appendix. ■

4.3.2 Homogeneous Battles

We further study the comparative statics when battles are homogeneous. In this case, players are homogeneous within each team and discriminatory powers across battles remain the same, while the two teams can be asymmetric.

When two teams are close to symmetry, i.e., p_A is close to 0.5, the minimum winning requirement in the majority rule with a headstart equals $K_S = \lfloor \frac{N}{2} \rfloor + 1 = \lfloor \frac{N}{2} + 1 \rfloor$. Hence, the conventional best-of-N allocation rule is optimal when the number of battles is odd, as summarized in Proposition 8.

Proposition 8 *With homogeneous battles, when two teams are close to symmetry, i.e., $p_A \in [\frac{1}{2}, \frac{N+1}{2N})$, the minimum winning requirement $K_S = \lfloor \frac{N}{2} + 1 \rfloor$. As a result, if the number of battles is odd, the simple majority rule is optimal.*

In a 3-battle (5-battle) team contest, best-of-three (best-of-five) is optimal when p_A is lower than 66.67% (60%). If the winning probability of the stronger team in each battle exceeds 66.67% (60%) with 3 (5) battles, a simple majority rule is no longer optimal.

We next turn to the unanimous rule. Intuitively, the unanimous rule extremely favors the weaker team. From the perspective of optimal favoritism, the unanimous rule would be optimal if and only if p_A is sufficiently close to 1, i.e., when the two teams are sufficiently asymmetric. This is confirmed by the following proposition.

Proposition 9 *With homogeneous battles, when two teams have a large disparity in strength, i.e., $p_A \in (\frac{N-1}{N}, 1)$, the minimum winning requirement $K_S = N$, i.e., the unanimous rule is optimal.*

Example. Consider a 3-battle team contest and each battle is modeled as a standard lottery contest. We plot the expected total efforts that result from the simple majority rule (MR, the lower curve) and the majority rule with optimal headstart (MRH, the higher curve) in Figure 2 by varying c_A while fixing $c_B = 1$.

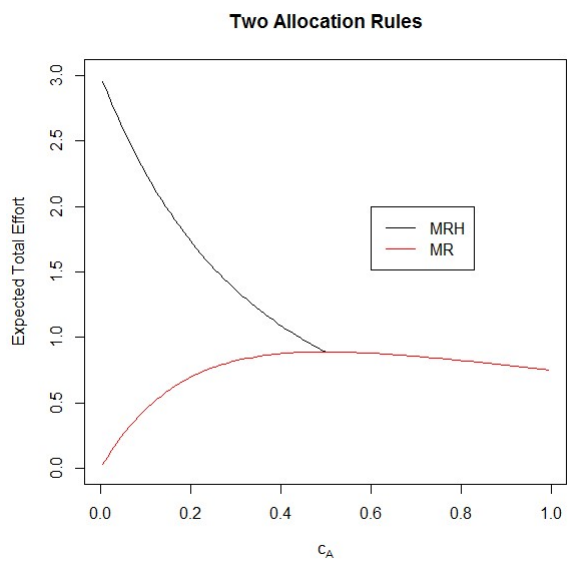

Figure 2: Comparisons

By Proposition 8, when the two teams are close to symmetry, the optimal allocation rule converges to the simple majority rule. In Figure 2, the two curves merge and coincide when c_A is sufficiently close to $c_B = 1$. On the contrary, when the two teams become sufficiently asymmetric, the unanimous rule becomes optimal and outperforms the simple majority rule. Note that with the simple majority rule, the total effort does not change monotonically when

c_A increases. However, under the optimal design, the total effort always increases when team A gets stronger. Clearly, leveling the playing field significantly enhances effort supply when the teams get more asymmetric.

4.4 Extensions

To check the robustness of the insights from Section 3 and gain new insights in different settings, we explore a number of extensions, including maximizing the effort of the winning team, permitting negative prizes, and relaxing budget balance constraints. These modifications result in a breakdown of the linearity of the organizer's problem and render the linear programming method ineffective. Due to the complexity of the non-linearity issue, we use numerical simulations to analyze 2-battle or 3-battle team contests with homogeneous battles in different extensions. Moreover, we assume each battle is a standard lottery contest and fix $c_B = 1$. In general, our simulations show that even for homogeneous battles, path-dependent prizes or partial prizes may arise at the optimum in these extensions.

4.4.1 Maximizing the Winning Team's Effort

According to the history independence result, the design of v^A does not disrupt $\Pr(\mathcal{W}^A)$ for all $\mathcal{W}^A \in 2^{\mathcal{N}}$ even after adjusting for trivial battles. As a consequence, the (ex-ante) expected winner's effort can be written as

$$\mathbf{WE}(v^A) \triangleq \sum_{\mathcal{W}^A \subseteq \mathcal{N}} \Pr(\mathcal{W}^A) \left[v^A(\mathcal{W}^A) \sum_{t \in \mathcal{N}} \mathbb{E}x_{A(t)}(\mathcal{W}^A \cap \mathcal{N}_t) + \big(1 - v^A(\mathcal{W}^A)\big) \sum_{t \in \mathcal{N}} \mathbb{E}x_{B(t)}(\mathcal{W}^A \cap \mathcal{N}_t) \right].$$

Here, $v^A(\mathcal{W}^A) \sum_{t \in \mathcal{N}} \mathbb{E}x_{A(t)}(\mathcal{W}^A \cap \mathcal{N}_t) + \big(1 - v^A(\mathcal{W}^A)\big) \sum_{t \in \mathcal{N}} \mathbb{E}x_{B(t)}(\mathcal{W}^A \cap \mathcal{N}_t)$ is the expected winner's effort given a history $\mathcal{W}^A$, where $v^A(\mathcal{W}^A)$ is interpreted as the winning odd for team A to win an indivisible prize of value 1. It is straightforward to check that $\mathbf{WE}(v^A)$ is a quadratic function of $v^A(\mathcal{W}^A)$. Thus, we can no longer expect that the optimal prize allocation involves no split prizes in general, since the designer's problem is not a linear program anymore.[24]

[24] If contest technologies are not homogeneous of degree zero, the total effort is also a non-linear function of the prizes. One can reasonably expect that the findings in this subsection would apply similarly. For

Consider a 3-battle team contest with each battle as a standard lottery and $c_B = 1$. Our simulations show that the winner remains the same in five out of eight possible paths as p_A or c_A varies in the optimal design. More specifically, $v^A(\varnothing) = v^A(\{1\}) = v^A(\{2\}) = v^A(\{3\}) = 0$ and $v^A(\{1,2,3\}) = 1$. These results say that team A never wins the contest if it wins no more than one battle, and it always wins the contest if it wins all battles.

Moreover, when c_A is above roughly 0.475, the simple majority rule is optimal, and when c_A is below roughly 0.345, the unanimous rule favoring team B is optimal. These rules are path-independent and involve no split prizes. In contrast, when c_A is in between the above two cutoffs, the optimal rule is path-dependent. In Figure 3, we plot the values of $v^A(\{1,2\}), v^A(\{1,3\}), v^A(\{2,3\})$ for the optimal rule while allowing path dependence as c_A changes within the interval $[0.33, 0.5]$. Clearly, split prizes are often involved at optimum as $v^A(\{1,2\})$ typically falls in $(0,1)$.

Figure 4 plots the improvement of the optimal path-dependent rule compared with the optimal path-independent rule. Within the class of path-independent rules, when c_A is above roughly 0.425 (see the dashed vertical line in Figure 4), the simple majority rule is optimal, and when c_A is below 0.425, the unanimous rule favoring team B is optimal. We thus must have $v^A(0) = v^A(1) = 0$, and $v^A(2)$ is described in Figure 4.

Figure 4 reveals that, at the cutoff (roughly 0.425) of c_A at which the optimal path-independent rule switches from the simple majority rule to the unanimous rule, the optimal path-dependent rule outperforms the optimal-path-independent rule to a great extent (see the peak of the curve in Figure 4).

When the two teams are sufficiently symmetrical (i.e., c_A gets high enough), the optimal path-independent allocation rule is the simple majority rule; when the two teams are sufficiently asymmetric (i.e., c_A gets low enough), the unanimous rule would be optimal and outperforms the simple majority rule. Hence, the primary insight of this paper preserves even when the objective changes to maximizing the winner's effort: The optimal design levels the playing field in an asymmetric team contest.

Nevertheless, our exercise reveals that, in general, even with homogeneous battles, max-

example, consider two homogeneous battles, each with a ratio form success function $p_{i(t)}(x_{A(t)}, x_{B(t)}) = \frac{f(x_{i(t)})}{f(x_{A(t)})+f(x_{B(t)})}$ where $f(x) = x + 0.2$. All players' marginal costs are 1. We find that the optimal prize rule is to assign the entire prize to an arbitrarily fixed battle. This rule is path-dependent.

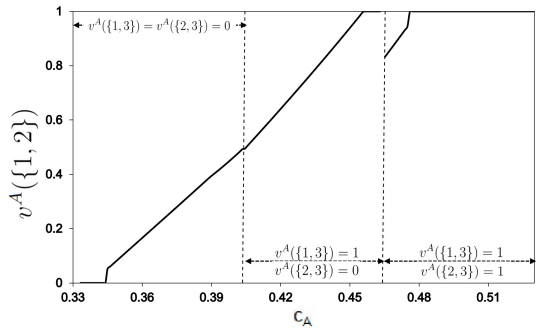

Figure 3: Optimal Path-dependent Rule

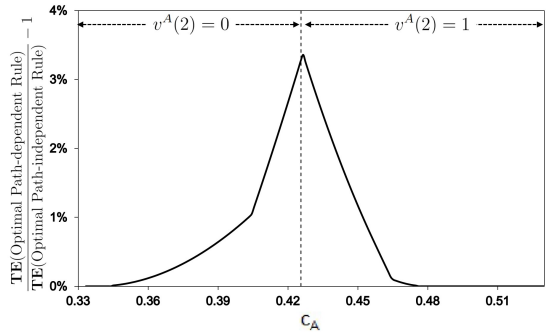

Figure 4: Improvement of Path-dependent Rule

imizing the winning team's effort would necessarily involve path-dependent prizes and split prizes, which does not occur for total effort maximization.

4.4.2 Relaxing the Nonnegativity Condition

The prohibition of negative prizes permits the elimination of individual rationality constraints from the optimization problem. However, when negative prizes are allowed, we need to ensure that players are willing to participate in the team contest by taking into account their individual rationality conditions.[25] That is,

$$\mathbb{E}\{u_{i(t)}(v^A)\} \geq 0, \forall i \in \{A, B\}, t \in \mathcal{N}, \tag{IR}$$

where $\mathbb{E}u_{i(t)}(v^A) \triangleq \mathbb{E}_{\mathcal{W}^A} v^i(\mathcal{W}^A) - c_{i(t)}\alpha_{i(t)}\mathbf{PS}_t(v^A)$ denotes the (ex-ante) expected payoff of player $i(t)$ given v^A. Note that $\mathbb{E}_{\mathcal{W}^A} v^i(\mathcal{W}^A)$ represents the expected gain from winning the prizes, and $c_{i(t)}[\alpha_{i(t)}\mathbf{PS}_t(v^A)]$ is the expected effort cost in the battle t.

With full-fledged heterogeneity, solving the optimal design is technically difficult since Lemma 3 (optimality of 0 or 1 prizes) fails. In particular, it is challenging to characterize the implications of vertex rules when negative prizes are allowed. With homogenous battles, any path-dependent prize allocation rule can be duplicated by a path-independent one. Given a specific number of winning battles, the constructed path-independent rule averages the prizes in a path-dependent rule for winning the concerned number of battles. Therefore, there always exists an optimal rule being path-independent.

[25] We assume that each player has to decide whether to participate at the start of the contest.

More importantly, the optimal path-independent rule may not be unique and the optimized total effort level must be $\frac{N\alpha}{c_A\alpha_A+c_B\alpha_B}$, where α_i represents the ratio of effort to the prize spread for a player on team $i \in \{A, B\}$ and $\alpha = \alpha_A + \alpha_B$.[26]

We next conduct a series of numerical simulations to further illustrate the above points. In a 3-battle team contest with homogeneous standard lottery battles and $c_B = 1$, the optimal path-independent rule is typically not unique and the resultant highest total effort level induced by the optimal rule is $\frac{3c_A+3}{2c_A}$. In particular, if $c_A = 0.5$, then the three prize allocation rules shown in Table 2 yield the same level of highest expected total effort.

Table 2: Three Equivalent Rules ($c_A = 0.5$)

	$v^A(0)$	$v^A(1)$	$v^A(2)$	$v^A(3)$
Rule 1	-3.25	-3.25	1.8125	1.8125
Rule 2	-19	1.25	1.25	1.25
Rule 3	-1	-1	-1	4.0625

Moreover, we can always construct an optimal path-independent rule with $v^A(0) = v^A(1)$ and $v^A(2) = v^A(3)$ for an arbitrary $c_A \in (0, 1]$. Adopting this class of rules, Figure 5 shows that (i) the optimal rule always favors the weaker team since $v^A(3) < 1 - v^A(0) = v^B(3)$ for all $c_A < 1$; and (ii) negative prizes are always beneficial as revealed by the highest curve of effort ratio.[27] Allowing penalties in team contests makes the effort level at least 4 times the original level. The lower bound 4 is achieved if and only if $c_A = 1$, where $v^A(0) = -1.5$ and $v^A(3) = 2.5$.

4.4.3 Relaxing the Budget Balance Condition

For tractability, we consider a 2-battle team contest with homogeneous battles. In this case, there are four possible outcomes, $\mathcal{W}^A \in \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$ and $\mathcal{W}^B = \{1, 2\} \setminus \mathcal{W}^A$. The prize allocation rule should specify eight values, $v^i(\varnothing), v^i(\{1\}), v^i(\{2\}), v^i(\{1, 2\}),\ i \in$

[26] We first realize that individual rationality constraints must be binding at optimum. Namely, $\mathbb{E}_{\mathcal{W}^A} v^i(\mathcal{W}^A) = c_i\alpha_i \mathbf{PS}(v^A), i \in \{A, B\}$, where $\mathbf{PS}(v^A)$ denotes the prize spread for every battle. Hence, the optimal rule satisfies that $\frac{\mathbb{E}_{\mathcal{W}^A} v^A(\mathcal{W}^A)}{1-\mathbb{E}_{\mathcal{W}^A} v^A(\mathcal{W}^A)} = \frac{c_A\alpha_A}{c_B\alpha_B}$, which indicates $\mathbb{E}_{\mathcal{W}^A} v^A(\mathcal{W}^A) = \frac{c_A\alpha_A}{c_A\alpha_A+c_B\alpha_B}$. Therefore, we have $\mathbf{PS}(v^A) = \frac{1}{c_A\alpha_A+c_B\alpha_B}$ and $\mathbf{TE}(v^A) = \frac{N\alpha}{c_A\alpha_A+c_B\alpha_B}$. Hence, v^A is optimal as long as it satisfies (i) $\mathbb{E}_{\mathcal{W}^A} v^A(\mathcal{W}^A) = \frac{c_A\alpha_A}{c_A\alpha_A+c_B\alpha_B}$, (ii) $\mathbf{PS}(v^A) = \frac{1}{c_A\alpha_A+c_B\alpha_B}$, (iii) monotonicity conditions, and (iv) budget-balance conditions. Typically, the above conditions do not pin down a unique solution for $N + 1$ unknowns.

[27] For an easier exposition, we drop the cases wherein $c_A \in (0, 0.3]$ to avoid the extremes. When $c_A = 0.1$, $v^A(0) = -35.3$; when $c_A = 0.01$, $v^A(0) = -2600.3$.

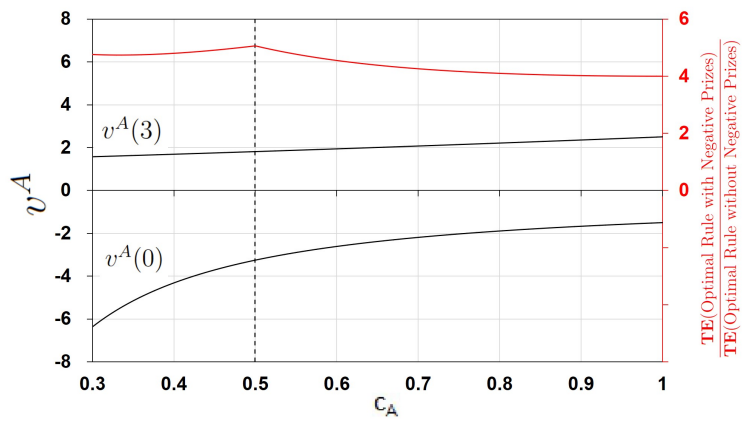

Figure 5: Negative Prizes are Beneficial

$\{A, B\}$. We still keep nonnegativity and monotonicity conditions in Assumption 2. However, the budget conditions now become:

$$v^i(\mathcal{W}^i) + v^j(\mathcal{W}^j) \leq 1, \forall \mathcal{W}^i, \mathcal{W}^j = \mathcal{N} \backslash \mathcal{W}^i, i, j \in \{A, B\}, i \neq j.$$

Let $p_{i(t)}(\tilde{v}_A, \tilde{v}_B)$ denote the winning probability of player $i(t)$ in the battle t when the effective prize spreads for player $A(t)$ and $B(t)$ are $\tilde{v}_A$ and $\tilde{v}_B$. Let $\mathbf{TE}_t(\tilde{v}_A, \tilde{v}_B)$ denote the total effort exerted in battle t. If team A wins the first battle, the effective prize spreads of player $A(2)$ and $B(2)$ are $V_{A(2)}(\{1\}) \triangleq v^A(\{1,2\}) - v^A(\{1\})$ and $V_{B(2)}(\varnothing) \triangleq v^B(\{2\}) - v^B(\varnothing)$. The expected total effort equals $\mathbf{TE}_2^A \triangleq \mathbf{TE}_2(V_{A(2)}(\{1\}), V_{B(2)}(\varnothing))$. If team B wins the first battle, the effective prize spreads of player $A(2)$ and $B(2)$ are $V_{A(2)}(\varnothing) \triangleq v^A(\{2\}) - v^A(\{\varnothing\})$ and $V_{B(2)}(\{1\}) \triangleq v^B(\{1,2\}) - v^B(\{1\})$, respectively. The expected total effort equals $\mathbf{TE}_2^B \triangleq \mathbf{TE}_2(V_{A(2)}(\varnothing), V_{B(2)}(\{1\}))$.

Consider the first battle. If $A(1)$ wins, his expected prize would be

$$\mathbb{E}(\text{A(1) wins}) \triangleq v^A(\{1,2\}) p_{A(2)}(V_{A(2)}(\{1\}), V_{B(2)}(\varnothing)) + v^A(\{1\}) p_{B(2)}(V_{A(2)}(\{1\}), V_{B(2)}(\varnothing)).$$

If $A(1)$ loses, his expected prize would be

$$\mathbb{E}(\text{A(1) loses}) \triangleq v^A(\{2\}) p_{A(2)}(V_{A(2)}(\varnothing), V_{B(2)}(\{1\})) + v^A(\varnothing) p_{B(2)}(V_{A(2)}(\varnothing), V_{B(2)}(\{1\})).$$

The effective prize spread of player $A(1)$ thus equals $V_{A(1)} \triangleq \mathbb{E}(\text{A(1) wins}) - \mathbb{E}(\text{A(1) loses})$. The effective prize spread $V_{B(1)}$ of player $B(1)$ can be defined analogously. Hence, the expected total effort exerted in the first battle is $\mathbf{TE}_1 \triangleq \mathbf{TE}_1(V_{A(1)}, V_{B(1)})$. Therefore, the designer's objective function can be expressed as

$$\mathbf{TE}(v^A, v^B) \triangleq \mathbf{TE}_1 + p_{A(1)}(V_{A(1)}, V_{B(1)})\mathbf{TE}_2^A + p_{B(1)}(V_{A(1)}, V_{B(1)})\mathbf{TE}_2^B.$$

In the following numerical exercise, we maintain the assumption that each battle is a standard lottery contest, fix $c_B = 1$, and allow c_A to vary within the interval $[0.5, 1]$.

With the budget balance condition, by Proposition 2, we can show that the following rule is optimal regardless of c_A across all path-dependent rules: $v^A(\varnothing) = v^A(\{1\}) = v^A(\{2\}) = 0, v^A(\{1,2\}) = 1$, and $v^B(\mathcal{W}^B) = 1 - v^A(\{1,2\} \setminus \mathcal{W}^B)$.

If we drop the budget balance condition, we find that $v^A(\varnothing) = v^B(\varnothing) = 0$ and $v^A(\{1,2\}) = v^B(\{1,2\}) = 1$ always hold for the optimal path-dependent rule; and $v^A(0) = v^B(0) = 0$, $v^A(2) = v^B(2) = 1$ always hold for the optimal path-independent rule. Moreover, we have $v^i(\{1\}) = 0$ in the optimal path-dependent rule for $i = A, B$. It remains to determine the values of $v^A(\{2\})$, $v^B(\{2\})$ in the optimal path-dependent rule and $v^A(1)$, $v^B(1)$ in the optimal path-independent rule. Based on numerical solutions, Figure 6 plots $v^A(\{2\})$, $v^B(\{2\})$ and $v^A(1)$, $v^B(1)$ as c_A varies.[28] Both the optimal path-independent and path-dependent cases involve partial prizes, and they are generally different, which means that the optimal rule is in general path-dependent even when the battles are homogeneous. This result further implies that the budget must be slack in general at optimum.

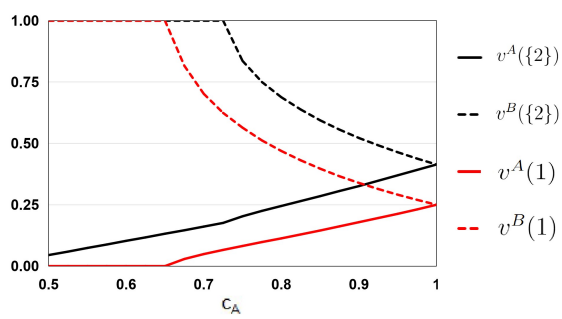

Figure 6: Optimal Path-(in)dependent Rules

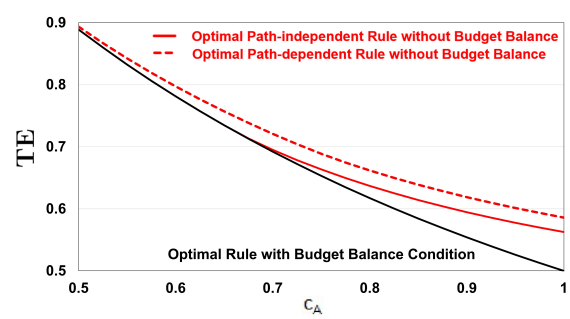

Figure 7: Comparisons

[28] When $c_A = 1$, $v^A(1) = v^B(1) = 0.25$ and $v^A(\{2\}) = v^B(\{2\}) = \sqrt{2} - 1$.

Figure 6 shows that the optimal rule is no longer path-independent. The optimal design of the competition only awards a positive prize to the team that wins the second battle. If this team also wins the first battle, it receives the full prize. However, if it loses the first battle, it still has a chance to win a slightly lower prize by winning the second battle. In that sense, the first battle can be interpreted as a warm-up for the second battle that determines the winner. As a result, the first two battles play different roles in prize allocations. However, this novel channel to incentivize competitors no longer works when the budget balance condition is binding.

We are now able to draw the total efforts under the three different rules mentioned above as c_A varies, in Figure 7. The effort supply under the optimal path-dependent rule without budget balance is weakly higher than that under the optimal path-independent rule without budget balance, which in turn is weakly higher than that under the optimal path-dependent rule with budget balance. This shows the benefit to the contest organizer of dropping the budget balance condition. It further confirms that the optimal design must be path-dependent when dropping the budget balance condition, even when the battles are homogeneous. As the two teams become more evenly matched, dropping the budget balance condition becomes increasingly beneficial. In particular, even when the two teams are completely symmetric (i.e., $c_A = 1$), relaxing the budget balance condition benefits the contest designer. When $c_A \leq \sqrt{2} - 1$, the budget balance condition is binding for optimal designs under all three scenarios. Therefore, three curves in Figure 7 coincide.

4.5 Implications

In various contexts of multi-battle team contests such as sports, R&D competitions, and political campaigns, designers are concerned with the aggregate productive effort of all members from both teams. Our results shed light on the effort-maximizing prize design of these contests.

Our study demonstrates that when two teams are more or less evenly matched, the simple majority rule is optimal. This rule is pervasive in top-notch sports contests with team titles wherein prizes are usually awarded following the best-of-5 rule, such as the Davis Cup and the Billie Jean King Cup in tennis; the Thomas Cup, the Uber Cup, and the Sudirman Cup

in badminton; and the Swaythling Cup and the Corbillon Cup in table tennis. Since the contending teams In these competitions are often of similar strength, our results provide a theoretical rationale to support the use of simple majority rules in these real-world team competitions.

Our research also provides insight into the design of legislative elections, where candidates from opposing parties vie for seats in each constituency. Typically, a party winning a simple majority of seats can form a government or set political agendas in the legislature.[29] Despite its advantages, this prevailing election rule may not be fair due to the incumbency advantage, which presumably generates deleterious effects on social welfare.[30] For example, an incumbency advantage could deter both parties from exerting consistent efforts that are essential for maintaining a well-functioning political system. This situation can be worsened, since the dominant party in power can further utilize gerrymandering to secure victories, which even gives an advantage to the stronger team in legislative elections. Our analysis suggests that a headstart should be granted to the challenging party (typically the weaker party) to elicit a more productive effort supply from all parties. Our analysis provides a theoretical foundation for the Independent Redistricting Commission (IRC) to serve its intended role of eliminating gerrymandering and promoting a more equitable political campaign.

In team competitions such as patent races, major grant competitions, and government procurements, competitions are held between research alliances. These alliances consist of member entities that specialize in different tasks, which enables the alliance to compete as a unified entity. For instance, in a major project of NSFC, there may be five topics designated for investigation, and two research alliances consisting of five universities or institutes each typically focus on its own expertise. Each member takes charge of one topic and competes with its counterpart from the rivaling alliance on the same task. The performance of each member on their assigned task, relative to their opponent, will affect the overall performance

[29]Some empirical studies show that gaining a majority of seats in the legislature is advantageous for the ruling party. For example, Cox and Magar (1999) evaluate the majority status in terms of contributions from political action committees by investigating changes in party control of the House and Senate. Snyder (1989) takes maximizing the probability of obtaining a majority status as a political party's objective.

[30]Incumbency advantage is a commonly research topic in studies of congressional elections, see, e.g., Levitt (1994) and Jamie, Engstrom, and Roberts (2007). Pastine and Pastine (2012) identify various channels through which incumbency advantages may harm social welfare. For example, the ruling party can be less productive in serving the public interest if it has a low risk of being defeated.

of the entire alliance. Our analysis suggests that the designer should take into account the diverse strengths of research alliances, which are multi-dimensional, and convert them into a single-dimensional strength measure of scores. In addition, the designer should utilize properly designed score-based prize allocation rules to favor the weaker team.

Our analysis also illustrates how the optimal design should take care of heterogeneity in productivity efficiencies across battles, or in contest organizer's values of efforts generated from different battles. In R&D contests with multiple stages or dimensions, each component battle may exhibit varying levels of productivity, referred to as heterogeneity across component battles. Our study recommends that in order to maximize total R&D effort, the organizer should assign a higher score to the stage that has higher research productivity.

In reality, the contest designer often values players' efforts differently across component battles. Our study suggests that the score assigned to a battle should be proportional to the weight that the designer places on the concerned battle's effort. In particular, if a battle's effort is more highly valued, it should be assigned a higher score, all other things being equal.

5 Concluding Remarks

This paper studies the effort-maximizing prize design in team contests with an arbitrary number of pairwise battles.[31] We incorporate *full-fledged heterogeneity* in our analysis, meaning that all players can be heterogeneous and contest technologies can differ across battles. The organizer is able to reward teams according to the full history subject to budget balance constraints. We find that the history independence result shown by Fu, Lu, and Pan (2015) still applies, i.e., the outcomes of early battles do not distort the winning probabilities in future battles. As a result, players' winning odds in each battle can be viewed as independent lotteries at any state, which ensures that the optimization problem can be solved using linear programming techniques.

We then derive the closed-form optimal prize allocation rule, which is a majority-score rule with a headstart score for the weaker team. Specifically, two teams collect scores by

[31] Effort or performance maximization is a common goal for contest design in the literature, such as Olszewski and Siegel (2020).

winning component battles, and they obtain the same score if they win the same battle. The scores can be different across battles. The rule favors the weaker team by awarding it a headstart score, and the team that accumulates higher total scores wins the whole prize.

Our study reveals an interesting connection between our optimal design and Elo rating. Each team's Elo rating change is the sum of Elo rating changes of its members. The sum of the two teams' Elo ratings remains constant. Our optimal rule can be implemented as follows: The team with an improved Elo rating would win the competition and collect the entire prize. Instead of ranking the individual agents, in our paper we utilize the Elo rating method to determine the winning team by aggregating ratings over the team members.

Our general procedure for deriving the optimal design still works when the designer values the effort across battles or players differently, the designer only values the higher effort in each battle. Analogous to Fu, Lu, and Pan (2015), we can easily show that our optimal designs are fully robust to incomplete information within battles and contest temporal structures.

We have included a number of extensions to check the robustness of the insights from the main setup, and gain new insights in different settings. New complications would arise when we relax the budget balance condition or non-negativity of prizes, or consider the maximization of the winner's effort as in Barbieri and Serena (2019). We rely on numerical exercises to investigate the implications of these issues in this paper. In general, even for homogeneous battles, path-dependent prizes or partial prizes may arise at the optimum.[32]

[32]There are alternative objectives beyond the maximization of total effort or winning team's effort. For example, Ely, Frankel, and Kamenica (2015) consider the maximization of suspense or surprise, which is measured by the conditional probability of outcomes in each stage. In our team contest framework, the conditional probability of an outcome in a specific battle is not influenced by the prize allocation rule.

Appendix

This appendix covers the proofs of Lemma 1, Lemma 2, Lemma 3, Theorem 1, Proposition 3, Proposition 4, and Proposition 7.

Proof of Lemma 1

If battle t is trivial, i.e., $U_t^A(\mathcal{W}_t^A \cup \{t\}) = U_t^A(\mathcal{W}_t^A)$, it elicits zero effort and its winning outcome is determined by the default tie-breaking rule; hence $U_{t-1}^A(\mathcal{W}_t^A) = U_t^A(\mathcal{W}_t^A \cup \{t\}) = U_t^A(\mathcal{W}_t^A)$. Therefore, the recursive definition for U^A, $U_{t-1}^A(\mathcal{W}_t^A) = p_{A(t)} U_t^A(\mathcal{W}_t^A \cup \{t\}) + \left(1 - p_{A(t)}\right) U_t^A(\mathcal{W}_{t+1}^A)$, holds for any $p_{A(t)} \in [0, 1]$.

We next show that the outcome of a trivial battle does not affect the boundary conditions, which ensures that our formula for the effective prize spread remains valid when trivial battles are taken into account.

Lemma A.1 (Outcome Equivalence) *If battle $t|\mathcal{W}_t^A$ is trivial, then for all possible sets of winning battles of team A for the remaining battles, $\mathcal{Q} \subseteq \mathcal{N} \backslash \mathcal{N}_{t+1}$, $v^A(\mathcal{W}_t^A \cup \{t\} \cup Q) = v^A(\mathcal{W}_t^A \cup \mathcal{Q})$.*

We now explain why the above result must hold. By monotonicity conditions, $v^A(\mathcal{W}_t^A \cup \{t\} \cup \mathcal{Q}) \geq v^A(\mathcal{W}_t^A \cup \mathcal{Q})$ for all $\mathcal{Q} \subseteq \mathcal{N} \backslash \mathcal{N}_{t+1}$ and hence $V_t(\mathcal{W}_t^A) \geq 0$. It is worth noting that $V_t(\mathcal{W}_t^A) = 0$ implies that $v^A(\mathcal{W}_{t-1}^A \cup \{t\} \cup \mathcal{Q}) = v^A(\mathcal{W}_{t-1}^A \cup \mathcal{Q})$, $\forall \mathcal{Q} \subseteq \mathcal{N} \backslash \mathrm{N}_{t+1}$. In words, the subgames of a team contest are exactly the same, regardless of the outcome of the trivial battle. As a result, the trivial battle is inconsequential in determining the prize.

Since the outcome of a trivial battle does not affect the boundary conditions or recursive definitions of U^A, we have the following two remarks.

Remark A.1 (State Equivalence) *If battle $t|\mathcal{W}_t^A$ is trivial, for $\widetilde{t} \geq t$ and $\mathcal{Q} \subseteq \mathcal{N}_{\widetilde{t}+1} \backslash \mathcal{N}_{t+1}$, $(\mathcal{N}_{\widetilde{t}+1}, \mathcal{W}_t^A \cup \{t\} \cup \mathcal{Q})$ and $(\mathcal{N}_{\widetilde{t}+1}, \mathcal{W}_t^A \cup \mathcal{Q})$ are equivalent states. In other words, (i) the expected prize is identical, $U_{\widetilde{t}}^A(\mathcal{W}_t^A \cup \{t\} \cup \mathcal{Q}) = U_{\widetilde{t}}^A(\mathcal{W}_t^A \cup \mathcal{Q})$; (ii) the effective prize spread is identical, $V_{\widetilde{t}+1}(\mathcal{W}_t^A \cup \{t\} \cup \mathcal{Q}) = V_{\widetilde{t}+1}(\mathcal{W}_t^A \cup \mathcal{Q})$.*

Remark A.2 (Transition Probability Irrelevance) *If battle $t|\mathcal{W}_t^A$ is trivial, both the expected prize and effective prize spread for all battles will not change if the transition probability for these two subgames changes.*

Remark A.2 illustrates the fact that if a battle is trivial, then for two decision points representing two outcomes of this trivial battle, the total effort generated until the contest ends does not depend on which decision point to go to. All subsequent processes are exactly the same for these two decision points. Starting from these two points, two subgames are identical and the expected prize and effective prize spread remain the same when the transition probability of these two subgames changes. As a result, we can freely adjust the winning probability of trivial battles.

If all battles before battle t are nontrivial, the probability that history $\mathcal{W}_t^A$ occurs can be calculated by the multiplicative law of probability and hence given by $\prod_{j\in\mathcal{W}_t^A} p_{A(j)} \prod_{j\in\mathcal{N}_t\setminus\mathcal{W}_t^A}(1-p_{A(j)})$. If some of the battles are trivial, we can adjust the probabilities such that history $\mathcal{W}_t^A$ occurs with probability $\prod_{j\in\mathcal{W}_t^A} p_{A(j)} \prod_{j\in\mathcal{N}_t\setminus\mathcal{W}_t^A}(1-p_{A(j)})$.

Proof of Lemma 2

We first express $U_t^A(\mathcal{W}_t^A)$ and $V^A(\mathcal{W}_t^A)$ in terms of $\{v^A(\mathcal{W}^A)\}_{\mathcal{W}^A\in 2^{\mathcal{N}}}$.

(i) Determine the coefficient of $v^A(\mathcal{W}^A)$ in $U_t^A(\mathcal{W}_{t+1}^A)$.

Suppose the first t battles are finished with history $\mathcal{W}_{t+1}^A$. Consider an outcome $\mathcal{W}^A$ that is possible to achieve after history $\mathcal{W}_{t+1}^A$, i.e., $\mathcal{W}^A\cap\mathcal{N}_{t+1}=\mathcal{W}_{t+1}^A$; it follows from the multiplicative law of probability that the coefficient of $v^A(\mathcal{W}^A)$ in $U_t^A(\mathcal{W}_{t+1}^A)$ is

$$\prod_{j\in\mathcal{W}^A\setminus\mathcal{W}_{t+1}^A} p_{A(j)} \prod_{j\in(\mathcal{N}\setminus\mathcal{N}_{t+1})\setminus(\mathcal{W}^A\setminus\mathcal{W}_{t+1}^A)} (1-p_{A(j)})$$

where $\mathcal{N}\setminus\mathcal{N}_{t+1}=\{t+1,\cdots,N\}$ denotes the set of battles that are carried out after battle t. Then, $\mathcal{W}^A\setminus\mathcal{W}_{t+1}^A$ represents the set of winning battles of team A among $\mathcal{N}\setminus\mathcal{N}_{t+1}$ while $(\mathcal{W}^A\setminus\mathcal{W}_{t+1}^A)\setminus(\mathcal{W}^A\setminus\mathcal{W}_{t+1}^A)$ represents the set of losing battles of team A among $\mathcal{N}\setminus\mathcal{N}_{t+1}$.

Therefore, $U_t^A(\mathcal{W}_{t+1}^A)$ is a linear function of $v^A(\mathcal{W}^A)$, for any $\mathcal{W}^A\subseteq\mathcal{N}$,

$$U_t^A(\mathcal{W}_{t+1}^A)=\sum_{\mathcal{W}^A:\mathcal{W}^A\cap\mathcal{N}_{t+1}=\mathcal{W}_{t+1}^A}\prod_{j\in\mathcal{W}^A\setminus\mathcal{W}_{t+1}^A} p_{A(j)} \prod_{j\in(\mathcal{N}\setminus\mathcal{N}_{t+1})\setminus(\mathcal{W}^A\setminus\mathcal{W}_{t+1}^A)} (1-p_{A(j)})v^A(\mathcal{W}^A).$$

In addition, the coefficient of $v^A(\mathcal{W}^A)$ in $U_t^A(\mathcal{W}_{t+1}^A)$ is zero when $\mathcal{W}^A\cap\mathcal{N}_{t+1}\neq\mathcal{W}_{t+1}^A$, i.e.,

$\mathcal{W}^A$ is impossible to achieve after history $\mathcal{W}^A_{t+1}$.

In sum, given $\mathcal{W}^A_{t+1}$, if $\mathcal{W}^A \cap \mathcal{N}_{t+1} \neq \mathcal{W}^A_{t+1}$, the coefficient is zero; if $\mathcal{W}^A \cap \mathcal{N}_{t+1} = \mathcal{W}^A_{t+1}$, the coefficient of $v^A(\mathcal{W}^A)$ in $U^A_t(\mathcal{W}^A_{t+1})$ is $\prod_{j\in\mathcal{W}^A\setminus\mathcal{W}^A_{t+1}} p_{A(j)} \prod_{j\in(\mathcal{N}\setminus\mathcal{N}_{t+1})\setminus(\mathcal{W}^A\setminus\mathcal{W}^A_{t+1})}(1-p_{A(j)})$.

(ii) Determine the coefficient of $v^A(\mathcal{W}^A)$ in $V_t(\mathcal{W}^A_t)$.

Note that the values of V_t are determined by U^A_t, $V_t(\mathcal{W}^A_t) = U^A_t(\mathcal{W}^A_t \cup \{t\}) - U^A_t(\mathcal{W}^A_t)$. Let $\mathcal{Q} \subseteq \mathcal{N}\setminus\mathcal{N}_{t+1}$ denote the set of winning battles of team A within the remaining $N-t$ battles. Rearranging the recursive definition for V, we can get

$$V_t(\mathcal{W}^A_t) = \sum_{\mathcal{Q}:\mathcal{Q}\subseteq\mathcal{N}\setminus\mathcal{N}_{t+1}} \prod_{j\in\mathcal{Q}} p_{A(j)} \prod_{j\in(\mathcal{N}\setminus\mathcal{N}_{t+1})\setminus\mathcal{Q}} (1-p_{A(j)}) \left[v^A(\mathcal{W}^A_t \cup \{t\} \cup \mathcal{Q}) - v^A(\mathcal{W}^A_t \cup \mathcal{Q})\right].$$

In particular, if $\mathcal{W}^A \cap \mathcal{N}_t \neq \mathcal{W}^A_t$, then there exists no $\mathcal{Q}$ such that $\mathcal{W}^A = \mathcal{W}^A_t \cup \{t\} \cup \mathcal{Q}$ or $\mathcal{W}^A = \mathcal{W}^A_t \cup \mathcal{Q}$, and hence the coefficient of $v^A(\mathcal{W}^A)$ in $V_t(\mathcal{W}^A_t)$ is zero; otherwise, $\mathcal{W}^A \cap \mathcal{N}_t = \mathcal{W}^A_t$, and the coefficient is nonzero, which depends on the outcome of battle t as follows.

If winning battle t, i.e., $t \in \mathcal{W}^A$, the coefficient of $v^A(\mathcal{W}^A)$ in $V_t(\mathcal{W}^A_t)$ is

$$\prod_{j\in\mathcal{W}^A\setminus\mathcal{W}^A_{t+1}} p_{A(j)} \prod_{j\in(\mathcal{N}\setminus\mathcal{N}_{t+1})\setminus(\mathcal{W}^A\setminus\mathcal{W}^A_{t+1})} (1-p_{A(j)}), \text{ where } \mathcal{W}^A_{t+1} = \mathcal{W}^A_t \cup \{t\}.$$

If losing battle t, i.e., $t \notin \mathcal{W}^A$, the coefficient of $v^A(\mathcal{W}^A)$ in $V_t(\mathcal{W}^A_t)$ is

$$-\prod_{j\in\mathcal{W}^A\setminus\mathcal{W}^A_{t+1}} p_{A(j)} \prod_{j\in(\mathcal{N}\setminus\mathcal{N}_{t+1})\setminus(\mathcal{W}^A\setminus\mathcal{W}^A_{t+1})} (1-p_{A(j)}), \text{ where } \mathcal{W}^A_{t+1} = \mathcal{W}^A_t.$$

Until now, we have all the building blocks to determine the coefficient of $v^A(\mathcal{W}^A)$ in $\mathbf{PS}_t(v^A)$ for an arbitrary $t \in \mathcal{N}$. If $t \in \mathcal{W}^A$, the coefficient of $v^A(\mathcal{W}^A)$ in $\mathbf{PS}_t(v^A)$ is

$$\begin{aligned}
&\underbrace{\prod_{j\in\mathcal{W}^A_t} p_{A(j)} \prod_{j\in\mathcal{N}_t\setminus\mathcal{W}^A_t} (1-p_{A(j)})}_{\text{the coefficient of } V_t(\mathcal{W}^A_t) \text{ in } \mathbf{PS}_t(v^A)} \underbrace{\prod_{j\in\mathcal{W}^A\setminus\mathcal{W}^A_{t+1}} p_{A(j)} \prod_{j\in(\mathcal{N}\setminus\mathcal{N}_{t+1})\setminus(\mathcal{W}^A\setminus\mathcal{W}^A_{t+1})} (1-p_{A(j)})}_{\text{the coefficient of } v^A(\mathcal{W}^A) \text{ in } V_t(\mathcal{W}^A_t) \text{ when } t\in\mathcal{W}^A} \\
= &\prod_{j\in\mathcal{W}^A, j\neq t} p_{A(j)} \prod_{j\notin\mathcal{W}^A, j\neq t} (1-p_{A(j)}).
\end{aligned}$$

Similarly, if $t \notin \mathcal{W}^A$, the coefficient of $v^A(\mathcal{W}^A)$ in $\mathbf{PS}_t(v^A)$ that determined by battle t is

$$-\prod_{j\in\mathcal{W}^A, j\neq t} p_{A(j)} \prod_{j\notin\mathcal{W}^A, j\neq t} (1-p_{A(j)}).$$

Hence, the coefficient of $v^A(\mathcal{W}^A)$ in $\mathbf{PS}_t(v^A)$ is

$$(-1)^{\mathbf{1}(t\notin\mathcal{W}^A)} \prod_{j\in\mathcal{W}^A, j\neq t} p_{A(j)} \prod_{j\notin\mathcal{W}^A, j\neq t} (1-p_{A(j)}).$$

Proof of Lemma 3

Step 1: Suppose that $\widehat{v}^A(\cdot)$ and $\widetilde{v}^A(\cdot)$ satisfy nonnegativity, monotonicity, and budget balance conditions in Assumption 2. Apparently, the convex combination of $(\cdot)$ and $\widetilde{v}^A(\cdot)$, i.e., $v^A(\cdot) = \theta\widehat{v}^A(\cdot) + (1-\theta)\widetilde{v}^A(\cdot), \theta \in (0,1)$, satisfy nonnegativity, monotonicity, and budget balance conditions.

We then establish the linearity of $\mathbf{TE}(v^A)$. Namely, $\mathbf{TE}(v^A) = \theta\mathbf{TE}(\widehat{v}^A) + (1-\theta)\mathbf{TE}(\widetilde{v}^A)$, which holds directly by Equation (2).

Step 2: Suppose that v^A takes values other than zero or one. We can always find out $\widehat{v}^A(\cdot) \neq \widetilde{v}^A(\cdot)$ and $\theta \in (0,1)$ such that $v^A(\cdot) = \theta\widehat{v}^A(\cdot) + (1-\theta)\widetilde{v}^A(\cdot)$, implying that v^A is not a vertex of $\mathcal{V}^A$. Immediately, the number of vertices is finite since v^A at vertices can only take values of zero or one.

To be specific, let $\gamma_1(v^A) = \max\{v^A(\mathcal{W}^A) : v^A(\mathcal{W}^A) < 1\}$ denote the maximum value of $v^A(\cdot)$ excluding 1; and $\gamma_2(v^A) = \max\{v^A(\mathcal{W}^A) : v^A(\mathcal{W}^A) < \gamma_1(v^A)\}$ denote the maximum value that $v^A(\cdot)$ takes excluding 1 and $\gamma_1(v^A)$. Note that $\gamma_1(v^A) \in (0,1)$ by our assumption, while it is possible that $\gamma_2(v^A) = 0$. Let $\mathcal{H}(v^A) = \{\mathcal{W}^A : v^A(\mathcal{W}^A) = \gamma_1(v^A)\}$ denote the set of final outcomes $\mathcal{W}^A$ such that $v^A(\mathcal{W}^A) = \gamma_1(v^A)$. We define two allocation rules in the following:

$$\widehat{v}^A(\mathcal{W}^A) = \begin{cases} 1, & \text{if } \mathcal{W}^A \in \mathcal{H}(v^A), \\ v^A(\mathcal{W}^A), & \text{otherwise,} \end{cases} \quad \text{and} \quad \widetilde{v}^A(\mathcal{W}^A) = \begin{cases} \gamma_2(v^A), & \text{if } \mathcal{W}^A \in \mathcal{H}(v^A), \\ v^A(\mathcal{W}^A), & \text{otherwise.} \end{cases}$$

In addition, both $\widehat{v}^A(\cdot)$ and $\widetilde{v}^A(\cdot)$ satisfy nonnegativity, monotonicity, and budget balance

conditions. Therefore, $v^A(\cdot) = \theta\widehat{v}^A(\cdot) + (1-\theta)\widetilde{v}^A(\cdot)$ for $\theta = \frac{\gamma_1(v^A)-\gamma_2(v^A)}{1-\gamma_2(v^A)} \in (0,1)$.

Proof of Theorem 1

Our purpose is to prove the optimal v^A is given by

$$v^A(\mathcal{W}^A) = \begin{cases} 1, & \text{if } w^A(\mathcal{W}^A) > S_A, \\ 0, & \text{if } w^A(\mathcal{W}^A) < S_A, \\ 0 \text{ or } 1, & \text{if } w^A(\mathcal{W}^A) = S_A. \end{cases}$$

By Equation (2),

$$\begin{aligned}
&\mathbf{TE}(v^A) \triangleq \sum_{t\in\mathcal{N}} \alpha_t \mathbf{PS}_t(v^A) \\
= &\sum_{t\in\mathcal{N}} \alpha_t \sum_{\mathcal{W}^A\subseteq\mathcal{N}} \left[(-1)^{\mathbf{1}(t\notin\mathcal{W}^A)} \left(\prod_{j\in\mathcal{W}^A, j\neq t} p_{A(j)} \right) \left(\prod_{j\notin\mathcal{W}^A, j\neq t} (1-p_{A(j)}) \right) v^A(\mathcal{W}^A) \right] \\
= &\sum_{t\in\mathcal{N}} \alpha_t \sum_{\mathcal{W}^A\subseteq\mathcal{N}} \left[(-1)^{\mathbf{1}(t\notin\mathcal{W}^A)} \left(\prod_{j\in\mathcal{W}^A, j\neq t} \frac{p_{A(j)}}{1-p_{A(j)}} \right) \left(\prod_{\tilde{t}\in\mathcal{N}} (1-p_{A(\tilde{t})}) \right) \frac{1}{1-p_{A(t)}} v^A(\mathcal{W}^A) \right] \\
= &\left(\prod_{\tilde{t}\in\mathcal{N}} (1-p_{A(\tilde{t})}) \right) \sum_{t\in\mathcal{N}} \frac{\alpha_t}{1-p_{A(t)}} \sum_{\mathcal{W}^A\subseteq\mathcal{N}} \left[(-1)^{\mathbf{1}(t\notin\mathcal{W}^A)} \prod_{j\in\mathcal{W}^A, j\neq t} \frac{p_{A(j)}}{1-p_{A(j)}} v^A(\mathcal{W}^A) \right] \\
= &\beta \sum_{t\in\mathcal{N}} \widehat{\alpha}_t \widehat{\mathbf{PS}}_t(v^A),
\end{aligned}$$

where $\widehat{\mathbf{PS}}_t(v^A) \triangleq \sum_{\mathcal{W}^A\subseteq\mathcal{N}} \left[(-1)^{\mathbf{1}(t\notin\mathcal{W}^A)} \prod_{j\in\mathcal{W}^A, j\neq t} \frac{p_{A(j)}}{1-p_{A(j)}} v^A(\mathcal{W}^A) \right]$, $\widehat{\alpha}_t \triangleq \frac{\alpha_t}{1-p_{A(t)}}$, and $\beta \triangleq \prod_{t\in\mathcal{N}}(1-p_{A(t)})$.

By Lemma 3, it suffices to focus on the allocations, which only take 0 or 1. Consider such a rule v^A that satisfies $v^A(\widehat{\mathcal{W}^A}) = 1$ and $\sum_{t\in\widehat{\mathcal{W}^A}} w_t < S_A$ for some $\widehat{\mathcal{W}^A}$, where $w_t = \widehat{\alpha}_t / p_{A(t)}$ and $S_A = \sum_{t\in\mathcal{N}} \widehat{\alpha}_t$. Since v^A only take 0 or 1, there must exist a $\underline{\mathcal{W}}^A \subseteq \widehat{\mathcal{W}^A}$ such that $v^A(\underline{\mathcal{W}}^A) = 1$ and $v^A(\mathcal{W}^A) = 0$ for any $\mathcal{W}^A \subsetneq \underline{\mathcal{W}}^A$. We then construct a feasible rule $\widehat{v}^A$ in the following $\widehat{v}^A(\mathcal{W}^A) = \begin{cases} 0, & \text{if } \mathcal{W}^A = \underline{\mathcal{W}}^A, \\ v^A(\mathcal{W}^A), & \text{otherwise.} \end{cases}$ We will claim that $\widehat{v}^A$ dominates v^A in terms of total effort induced.

When $v^A(\underline{\mathcal{W}}^A)$ changes from 1 to 0, the change in $\widehat{\mathbf{PS}}_t(v^A)$ equals

$$\Delta\widehat{\mathbf{PS}}_t(v^A) = (-1)^{\mathbf{1}(t\in\underline{\mathcal{W}}^A)} \prod_{j\in\underline{\mathcal{W}}^A, j\neq t} \frac{p_{A(j)}}{1-p_{A(j)}},$$

and the change in $\mathbf{TE}(v^A)$ thus equals

$$\begin{aligned}
\Delta\mathbf{TE}(v^A) &= \beta\sum_{t\in\mathcal{N}} \widehat{\alpha}_t \Delta\widehat{\mathbf{PS}}_t(v^A) \\
&= \beta\left[\sum_{t\in\underline{\mathcal{W}}^A} \widehat{\alpha}_t \Delta\widehat{\mathbf{PS}}_t(v^A) + \sum_{t\notin\underline{\mathcal{W}}^A} \widehat{\alpha}_t \Delta\widehat{\mathbf{PS}}_t(v^A)\right] \\
&= \beta\left[-\sum_{t\in\underline{\mathcal{W}}^A} \widehat{\alpha}_t \prod_{j\in\underline{\mathcal{W}}^A, j\neq t} \frac{p_{A(j)}}{1-p_{A(j)}} + \sum_{t\notin\underline{\mathcal{W}}^A} \widehat{\alpha}_t \prod_{j\in\underline{\mathcal{W}}^A, j\neq t} \frac{p_{A(j)}}{1-p_{A(j)}}\right] \\
&= \beta\left(\prod_{j\in\underline{\mathcal{W}}^A} \frac{p_{A(j)}}{1-p_{A(j)}}\right)\left[-\sum_{t\in\underline{\mathcal{W}}^A} \widehat{\alpha}_t \frac{1-p_{A(t)}}{p_{A(t)}} + \sum_{t\notin\underline{\mathcal{W}}^A} \widehat{\alpha}_t\right] \\
&= \beta\left(\prod_{j\in\underline{\mathcal{W}}^A} \frac{p_{A(j)}}{1-p_{A(j)}}\right)\left[\sum_{t\in\underline{\mathcal{W}}^A} \widehat{\alpha}_t + \sum_{t\notin\underline{\mathcal{W}}^A} \widehat{\alpha}_t - \sum_{t\in\underline{\mathcal{W}}^A} \frac{\widehat{\alpha}_t}{p_{A(t)}}\right] \\
&= \beta\left(\prod_{j\in\underline{\mathcal{W}}^A} \frac{p_{A(j)}}{1-p_{A(j)}}\right)\left[S_A - \sum_{t\in\underline{\mathcal{W}}^A} w_t\right] \\
&\geq \beta\left(\prod_{j\in\underline{\mathcal{W}}^A} \frac{p_{A(j)}}{1-p_{A(j)}}\right)\left[S_A - \sum_{t\in\widehat{\mathcal{W}^A}} w_t\right] > 0,
\end{aligned}$$

where $w_t = \frac{\alpha_t}{p_{A(t)}p_{B(t)}}$ and $S_A = \sum_{t\in\mathcal{N}} \frac{\alpha_t}{1-p_{A(t)}}$. This implies that for an allocation rule v^A, if there exists a $\widehat{\mathcal{W}^A}$ such that $v^A(\widehat{\mathcal{W}^A}) = 1$ and $\sum_{t\in\widehat{\mathcal{W}^A}} w_t < S_A$, we can always construct a feasible rule $\widehat{v}^A$ that yields strictly greater expected total effort than v^A does, as a result, v^A is not optimal. Therefore, the optimal rule must satisfy that

$$v^A(\mathcal{W}^A) = 0 \text{ whenever } w^A(\mathcal{W}^A) < S_A.$$

Analogously, consider a rule v^A that takes 0 and 1. If there exists a $\widehat{\mathcal{W}^A}$ such that $v^A(\widehat{\mathcal{W}^A}) = 0$ and $\sum_{t\in\widehat{\mathcal{W}^A}} w_t > S_A$. As before, we can always find a $\overline{\mathcal{W}}^A \supseteq \widehat{\mathcal{W}^A}$ such that

$v^A(\overline{\mathcal{W}}^A) = 0$ and $v^A(\mathcal{W}^A) = 1$ for all $\mathcal{W}^A \supsetneqq \overline{\mathcal{W}}^A$. We construct a feasible rule $\widetilde{v}^A$ such that $\widetilde{v}^A(\mathcal{W}^A) = \begin{cases} 1 & \text{if } \mathcal{W}^A = \overline{\mathcal{W}}^A, \\ v^A(\mathcal{W}^A) & \text{otherwise.} \end{cases}$ We will claim that $\widetilde{v}^A$ dominates v^A in terms of the expected total effort induced.

When $v^A(\overline{\mathcal{W}}^A)$ changes from 0 to 1, the change in $\widehat{\mathbf{PS}}_t(v^A)$ equals

$$\Delta\widehat{\mathbf{PS}}_t(v^A) = (-1)^{\mathbf{1}(t \notin \overline{\mathcal{W}}^A)} \prod_{j \in \overline{\mathcal{W}}^A, j \neq t} \frac{p_{A(j)}}{1 - p_{A(j)}},$$

and the change in $\mathbf{TE}(v^A)$ thus equals

$$\begin{aligned} \Delta\mathbf{TE}(v^A) &= \beta \sum_{t \in \mathcal{N}} \widehat{\alpha}_t \Delta\widehat{\mathbf{PS}}_t(v^A) \\ &= \beta \left(\prod_{j \in \overline{\mathcal{W}}^A} \frac{p_{A(j)}}{1 - p_{A(j)}} \right) \left[\sum_{t \in \overline{\mathcal{W}}^A} w_t - S_A \right] \\ &\geq \beta \left(\prod_{j \in \overline{\mathcal{W}}^A} \frac{p_{A(j)}}{1 - p_{A(j)}} \right) \left[\sum_{t \in \widehat{\mathcal{W}^A}} w_t - S_A \right] > 0, \end{aligned}$$

where $w_t = \frac{\alpha_t}{p_{A(t)} p_{B(t)}}$ and $T_A = \sum_{t \in \mathcal{N}} \frac{\alpha_t}{1 - p_{A(t)}}$. This implies that for some allocation rule v^A, if there exists a $\widehat{\mathcal{W}^A}$ such that $v^A(\widehat{\mathcal{W}^A}) = 0$ and $\sum_{t \in \widehat{\mathcal{W}^A}} w_t > S_A$, we can always construct a feasible rule $\widetilde{v}^A$ that gives strictly greater expected total effort than v^A, as a result, v^A is not optimal. Therefore, the optimal rule must satisfy that

$$v^A(\mathcal{W}^A) = 1 \text{ whenever } w^A(\mathcal{W}^A) > S_A.$$

Up to now, it is shown that $v^A(\mathcal{W}^A) = \begin{cases} 1, & \text{if } w^A(\mathcal{W}^A) > S_A, \\ 0, & \text{if } w^A(\mathcal{W}^A) < S_A. \end{cases}$ It remains to investigate the case where $w^A(\mathcal{W}^A) = S_A$. Consider a $\mathcal{W}^A$ such that $w^A(\mathcal{W}^A) = S_A$. Clearly, both $v^A(\mathcal{W}^A) = 0$ and $v^A(\mathcal{W}^A) = 1$ are feasible, since monotonicity conditions hold. By previous analysis, the expected total effort remains unchanged when $v^A(\mathcal{W}^A)$ switches from 0 to 1 or from 1 to 0. The argument holds for all $\mathcal{W}^A$ such that $w^A(\mathcal{W}^A) = S_A$. We therefore complete our analysis by discussing all the three cases.

In particular, when there does not exist a $\mathcal{W}^A$ such that $w^A(\mathcal{W}^A) = S_A$, the optimal prize allocation rule is unique. Otherwise, the optimal prize allocation rule is not unique, since $v^A(\mathcal{W}^A)$ can take either 0 or 1 for any $\mathcal{W}^A$ such that $w^A(\mathcal{W}^A) = S_A$.

Proof of Proposition 3

When $r(t) = 1$, $s_{i(t)} = \frac{\alpha_t}{1-p_{i(t)}} = \frac{(c_{A(t)}+c_{B(t)})^{-1}}{c_{i(t)}/(c_{A(t)}+c_{B(t)})} = \frac{1}{c_{i(t)}}$. Since $w_t = \frac{1}{c_{A(t)}} + \frac{1}{c_{B(t)}}$, the battle score w_t increases as one player becomes more efficient for a lower marginal cost. In contrast, since $\Delta s_t = \frac{1}{c_{A(t)}} - \frac{1}{c_{B(t)}}$, the headstart score H increases as $c_{A(t)}$ decreases or $c_{B(t)}$ increases.

Proof of Proposition 4

Let $z = \frac{\min(c_{A(t)},c_{B(t)})}{\max(c_{A(t)},c_{B(t)})}$ and $\hat{r}(z) \in (1,2)$ represent the unique solution to $r = 1 + z^r$.

(i) When $r(t) \leq \hat{r}(z)$, $w_t = \frac{r(t)(c_{A(t)}+c_{B(t)})}{c_{A(t)}c_{B(t)}}$. Then, the battle score grows with the discriminatory power. Meanwhile, $\Delta s_t = \frac{r(t)(c_{B(t)}-c_{A(t)})}{c_{A(t)}c_{B(t)}}$. Thus, the headstart score grows with the discriminatory power if and only if $c_{A(t)} < c_{B(t)}$.

(ii) When $r(t) \in (\hat{r}(z), 2]$, $w_t = \frac{c_{A(t)}+c_{B(t)}}{c_{A(t)}c_B(t)} \frac{1}{1-\max(p_{A(t)},p_{B(t)})}$, which increases with $\max(p_{A(t)}, p_{B(t)})$. According to Equation (5), $\max(p_{A(t)}, p_{B(t)})$ increases with $r(t)$, the battle score increases with $r(t)$. Similarly, $\Delta s_t = \frac{c_{A(t)}+c_{B(t)}}{c_{A(t)}c_B(t)} \frac{p_{A(t)}-p_{B(t)}}{1-\max(p_{A(t)},p_{B(t)})}$. If $c_{A(t)} < c_{B(t)}$, the headstart score grows with $r(t)$; otherwise, the headstart score decreases with $r(t)$.

(iii) When $r(t) > 2$, both $s_{A(t)}$ and $s_{B(t)}$ are not affected by $r(t)$. Then, the battle score and the headstart score remain unchanged as $r(t)$ changes.

Proof of Proposition 7

Suppose that two players on team B, $B(t')$ and $B(t'')$, have higher winning probabilities than players $A(t')$ and $A(t'')$ on team A, respectively. Without loss of generality, we assume $w_{t'} \leq w_{t''}$. The winning threshold for team B must be larger than $w_{t'}$: $S_B = \sum_{t\in\mathcal{N}} p_{B(t)}w_t > p_{B(t')}w_{t'} + p_{B(t'')}w_{t''} \geq 0.5(w_{t'} + w_{t'}) \geq w_{t'}$. This implies that team B does not earn sufficient scores by merely winning battle t', which means that the unanimous rule is not optimal.

References

[1] Arbatskaya, M., and Konishi, H., 2023. Dynamic team contests with complementary efforts. *Review of Economic Design*, forthcoming.

[2] Baik, K.H., Kim, I.G., and Na, S., 2001. Bidding for a group-specific public-good prize. *Journal of Public Economics*, 82, 415–429.

[3] Barbieri, S., Malueg, D.A., and Topolyan, I., 2014. The best-shot all-pay (group) auction with complete information. *Economic Theory,* 57(3), 603–640.

[4] Barbieri, S. and Serena, M., 2019. Winners' efforts in team contests. *Working Paper*.

[5] Chowdhury, S.M., Lee, D., and Topolyan, I., 2016. The max-min group contest: Weakest-link (group) all-pay auction. *Southern Economic Journal*, 83(1), 105–125.

[6] Clark, D. and Nilssen, T., 2020. Creating balance in dynamic competitions, *International Journal of Industrial Organization*, 69, 102578.

[7] Cox, G. and Magar, E., 1999. How much is majority status in the U.S. congress worth? *American Political Science Review*, 93(2), 299–309.

[8] Crutzen, B.S.Y., Flamand S., and Sahuguet N., 2020. A model of a team contest, with an application to incentive under list proportional representation, *Journal of Public Economics*, 182, 104019.

[9] Eliaz, K. and Wu, Q., 2018. A simple model of competition between teams. *Journal of Economic Theory*, 176, 372–392.

[10] Ely, J., Frankel, A., and Kamenica, E., 2015. Suspense and surprise. *Journal of Political Economy*, 123(1), 215–260.

[11] Feng, X. and Lu, J., 2018. How to split the pie: optimal rewards in dynamic multi-battle competitions. *Journal of Public Economics*, 160, 82–95.

[12] Ferrall, C. and Smith Jr., A.A., 1999. A sequential game model of sports championship series: theory and estimation. *Review of Economics and Statistics*, 81(4), 704–719.

[13] Franke, J., Kanzow, C., Leininger, W., and Schwartz, A., 2013. Effort maximization in asymmetric contest games with heterogeneous contestants. *Economic Theory*, 52(2), 589–630.

[14] Fu, Q., Lu, J., and Pan, Y., 2015. Team contests with multiple pairwise battles. *American Economic Review*, 105(7), 2120–2140.

[15] Fu, Q. and Wu, Z., 2020. On the optimal design of biased contests. *Theoretical Economics*, 15(4), 1435–1470.

[16] Gauriot, R. and Page, L., 2019. Does success breed success? a quasi-experiment on strategic momentum in dynamic contests. *Economic Journal*, 129, 3107–3136.

[17] Gelder, A., 2014. From Custer to Thermopylae: last stand behavior in multi-stage contests. *Games and Economic Behavior*, 87, 442–466.

[18] Häfner, S., 2017. A tug-of-war team contest. *Games and Economic Behavior*, 104, 372–391.

[19] Häfner, S., 2022. Eternal peace in the tug-of-war? *Economic Theory*, 74(4), 1057–1101.

[20] Harris, C. and Vickers, J., 1987. Racing with uncertainty. *Review of Economic Studies*, 54(1), 1–21.

[21] Jamie, C., Engstrom, E., and Roberts, J., 2007. Candidate quality, the personal vote, and incumbency advantage in congress. *American Political Science Review*, 101(2), 290–301.

[22] Jiang, X., 2018. Relative performance prizes and dynamic incentives in best-of-N contests. *Review of Industrial Organization*, 53, 563–590.

[23] Klumpp, T. and Polborn, M.K., 2006. Primaries and the new Hampshire effect. *Journal of Public Economics*, 90(6–7), 1073–1114.

[24] Klumpp, T., Konrad, K.A., and Solomon, A., 2019. The dynamics of majoritarian Blotto games. *Games and Economic Behavior*, 117, 402–419.

[25] Konishi, H., Pan, C., and Simeonov, D., 2022. Equilibrium player choices in team contests with multiple pairwise battles. *Games and Economic Behavior*, 132, 274–287.

[26] Konrad, K.A. and Kovenock, D., 2009. Multi-battle contests. *Games and Economic Behavior*, 66(1), 256–274.

[27] Lemus, J. and Marshall, G., 2021. Dynamic tournament design: evidence from prediction contests. *Journal of Political Economy*, 129(2), 383-420.

[28] Lemus, J. and Marshall, G., 2022. Contingent prizes in dynamic contests. *Working Paper*.

[29] Levitt, S. and Wolfram, C., 1997. Decomposing the sources of incumbency advantage in the US House of Representatives. *Legislative Studies Quarterly*, 22(1), 45–60.

[30] Li, S. and Yu, J., 2012. Contests with endogenous discrimination. *Economics Letters*, 117(3), 834–836.

[31] Liu, B., Lu, J., Wang, R., and Zhang, J., 2018. Optimal prize allocation in contests: the role of negative prizes. *Journal of Economic Theory*, 175, 291–317.

[32] Liu, B. and Lu, J., 2023. Optimal orchestration of rewards and punishments in rank-order contests. *Journal of Economic Theory*, 208, 105594.

[33] Moldovanu, B., Sela, A., and Shi, X., 2012. Carrots and sticks: prizes and punishments in contests. *Economic Inquiry*, 50(2), 453–462.

[34] Olszewski, W. and Siegel, R., 2020. Performance-maximizing large contest. *Theoretical Economics*, 15(1), 57–88.

[35] Pastine, I. and Pastine, T., 2012. Incumbency advantage and political campaign spending limits. *Journal of Public Economics*, 96(1–2), 20–32.

[36] Seel, C. and Wasser, C., 2014. On optimal head starts in all-pay auctions. *Economics Letters*, 124(2), 211–214.

[37] Sela, A. and Tsahi, O., 2020. On the optimal allocation of prizes in best-of-three all-pay auctions. *Social Choice and Welfare*, 55, 255–273.

[38] Snyder, J.M., 1989. Election goals and the allocation of campaign resources. *Econometrica*, 57(3), 637–660.

[39] Topolyan, I., 2014. Rent-seeking for a public good with additive contributions. *Social Choice and Welfare*, 42, 465–476.