9 Causal Analysis II

We have seen how multiple regression can be used to estimate causal parameters. These parameters, however, are not directly observable or measurable. Instead, by using a causal model and the observed variances and covariances for a sample of cases, we are able to make inferences about the values of the causal parameters of the system. That is, we use statistical summaries of observable phenomena to make inferences about unobservable phenomena. For example, in the simplest causal model, we divide the covariance between X and Y by the variance of X to get the least-squares estimate of the structural coefficient.

In this chapter, we will look at this observable–unobservable dichotomy from the opposite direction. That is, we will see how the causal parameters of a system, along with the variances and covariances of exogenous variables (which are taken as givens), create all of the additional observed variances and covariances in a system. Once we have estimated the parameters of the model, we can use these estimated parameters to partition the observed variances and covariances into components that result from the various causal processes included in the model. To begin, we will learn rules for reading covariance equations directly from a path diagram. These equations express the covariance between two variables in terms of components that are due to such processes as direct causes, indirect causes, correlated causes, and common causes. We will then learn similar rules for reading variance equations from a path diagram, equations that also involve direct, indirect, and correlated effects. These equations will show, however, that generally it is not possible to allocate unambiguously the variance of an endogenous variable among the various variables that are causes of it. Furthermore, in general it is also not

possible to allocate explained variance between direct effects and indirect effects. However, reduced-form equations that ascribe all of the variances and covariances to exogenous variables may help to clarify these issues.

The Simplest Causal Model

Let us start with the estimated parameters of a simple bivariate model,

$$X \xrightarrow{\;b\;} Y \longleftarrow e$$

The slope b of this linear causal system represents the change in Y caused by a unit increase in X. Since changes in X are not restricted to unity but instead can take on any value ΔX, the change in Y caused by ΔX will be $\Delta Y = b\Delta X$. The origin of a change in X equal to ΔX can occur at many different points in the range of values of X. Let us now think of measuring changes in X and Y as deviations from their respective means, that is, $\Delta X = X - \overline{X}$ and $\Delta Y = Y - \overline{Y}$. When we think of the mean as the origin of changes in X and Y, the effect of X on Y is the amount that Y will be caused to change or deviate from its mean when X changes or deviates from its mean by $\Delta X = X - \overline{X}$. This can be seen when we write the linear equation for the model in terms of deviation scores, in which case the intercept will equal zero:

$$Y - \overline{Y} = b(X - \overline{X}) + e \qquad \textbf{(9.1)}$$

If X changes or deviates from its mean by $X - \overline{X}$, it will cause a change or deviation in Y of $\hat{Y} - \overline{Y} = b(X - \overline{X})$. Since deviation scores are used to define the variance of variables, we can use Equation 9.1 to determine how much variance in Y is caused by X. The sample variance of Y equals

$$s_Y^2 = \frac{\sum (Y - \overline{Y})^2}{n}$$

Substituting Equation 9.1 for $Y - \overline{Y}$, we get

$$s_Y^2 = \frac{\sum [b(X - \overline{X}) + e]^2}{n}$$

Squaring the expression in the numerator to the right of the summation operator gives

$$s_Y^2 = \frac{\sum [b^2(X - \overline{X})^2 + 2b(X - \overline{X})e + e^2]}{n}$$

$$= \frac{\sum b^2(X - \overline{X})^2 + \sum 2b(X - \overline{X})e + \sum e^2}{n}$$

$$= \frac{b^2\sum (X - \overline{X})^2}{n} + \frac{2b\sum (X - \overline{X})e}{n} + \frac{\sum e^2}{n}$$

Dividing the sums of squares and sums of products by n gives the following variances and covariances:

$$s_Y^2 = b^2s_X^2 + 2bs_{Xe} + s_e^2$$

Since the covariance between X and e, s_{Xe}, is equal to zero for the sample regression results and is assumed to be zero in the population model, the formula for the variance of Y reduces to

$$s_Y^2 = b^2s_X^2 + s_e^2 \qquad \textbf{(9.2)}$$

Equation 9.2 shows that the variance of Y has been decomposed into two components, the variance caused by the observed variable X and the variance caused by all the unobserved variables e (which may also include measurement error). The first component in Equation 9.2 is the "explained" variance, and the second component is the "unexplained" variance. Although we already knew that the variance could be divided into explained and unexplained components, this new expression for the explained variance is very informative. It shows that the variance in Y caused by X, $b^2s_X^2$, equals the squared estimate of the structural coefficient times the variance of X. Thus, the more X varies, the greater will be the variance in Y caused by X. Also, the greater the absolute value of the structural coefficient, the greater will be the variance in Y caused by X. The derivation of Equation 9.2 shows that the structural coefficient is squared because the variance in Y consists of squared changes or deviations around its mean.

The variances of X and e are variances of exogenous variables and thus are created by causes outside of this system. If any outside sources caused the variances of either X or e either to increase or decrease, this would cause an increase or a decrease in the variance of the endogenous variable Y, even if the structural coefficient b were to remain the same. This fact has important implications for the coefficient of determination, which is the explained variance in Y divided by the total variance in Y, or

$$r_{XY}^2 = \frac{b^2s_X^2}{b^2s_X^2 + s_e^2}$$

If the variance of X increased, the numerator would increase proportionately more than the denominator; thus, the proportion of variance in Y that is explained by X would increase *even though the effect of X (i.e., b) did not change.* If our sample of observations were taken over a restricted range of X (i.e., we did not have a representative sample of X), the variance of X would decrease and the coefficient of determination would also decrease; if the effect of X were truly linear throughout its range, however, b would still be the same in the restricted range of observations. These two possibilities show how the measure of the strength of association is dependent on a factor, the variance of X, that is independent of the causal effect of X on Y. In an analogous situation, outside sources might cause the variance of e to increase without any changes occurring in the structural coefficient or the variance of X. This would increase the denominator of the coefficient of determination (it would increase the variance of Y) and thus reduce the value of the strength of association, even though the actual effect of X and its variance have not changed. Thus, the *proportion* of variance explained by X (r^2) might decrease even though the absolute amount of variance that is explained or caused by X does not change. In sum, Equation 9.2 makes it clear how changes in the variances of exogenous variables (X and e) may create changes in r^2 even though the effect of X on Y remains the same.

It is also possible to derive a formula that shows how changes in X (i.e., $\Delta X = X - \overline{X}$) create covariance between X and Y. The covariance is

$$s_{XY} = \frac{\sum (X - \overline{X})(Y - \overline{Y})}{n}$$

Substituting Equation 9.1 for $Y - \overline{Y}$,

$$s_{XY} = \frac{\sum (X - \overline{X})[b(X - \overline{X}) + e]}{n}$$

$$= \frac{\sum [b(X - \overline{X})(X - \overline{X}) + e(X - \overline{X})]}{n}$$

$$= \frac{b\sum (X - \overline{X})^2 + \sum e(X - \overline{X})}{n}$$

$$= bs_X^2 + s_{eX}$$

Since the covariance between the error term and X equals zero,

$$s_{XY} = bs_X^2 \qquad \textbf{(9.3)}$$

Thus, the covariance equals the effect of X times the variance of X. Since $Y - \overline{Y}$ is not squared in the formula for a covariance, b is not squared in Equation 9.3. The greater the effect of X and the greater the variance of X, the greater will be the covariance between X and Y. In this case, all of the covariance between X and Y is created by X; there is no spurious component to the covariance. Notice that Equation 9.3 is just a rearrangement of the terms in the formula for the bivariate regression slope $b = s_{XY}/s_X^2$.

A Two-Equation Model

Covariances

We have derived formulas that show how the variance of a dependent variable and the covariance between the dependent variable and an independent variable are created in a two-variable causal model. When more than one independent variable is included in the equation or when we have a multi-equation model, the algebra for deriving such formulas becomes rather complex and tedious. The procedure is the same as for a two-variable model, however. We first write an equation for each endogenous variable in the model in terms of deviation scores, such as Equation 9.1. We then follow the same procedures used above to derive an equation for the variance of each endogenous variable and an equation for each of the covariances that involve one or more endogenous variables. We will not go through these algebraic derivations but instead will use some relatively simple rules for "reading" these equations directly from the path diagram for the causal system.

Figure 9.1 presents the same two-equation model previously shown in Figure 8.3. The roman letter e for the error terms indicates that we will be examining formulas in terms of the sample statistics rather than the population parameters.

The regression equations for each endogenous variable are

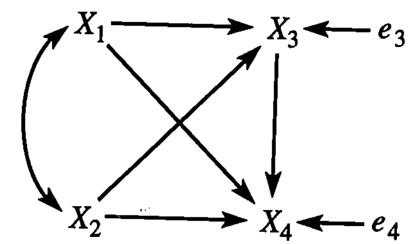
$$\hat{X}_3 = a_{3\cdot12} + b_{31}X_1 + b_{32}X_2$$

$$\hat{X}_4 = a_{4\cdot123} + b_{41}X_1 + b_{42}X_2 + b_{43}X_3$$

The regression coefficients, which are the estimates of the structural coefficients, could be placed on the appropriate paths to make Figure 9.1 complete.

Chain Rule for Covariances. Sewell Wright (1921) formulated a multiplication rule for reading each correlation from a diagram containing path (standardized) coefficients. The same rule may be used, with a slight modification, to read covariances from a diagram containing structural coefficients or unstandardized regression coefficients. The rule for determining the covariance s_{ij} involves finding each distinct chain that links X_i and X_j. Each chain has X_i at one end and X_j at the other end. To find each chain, read back from X_j to X_i, where X_j appears "later" in the model, along each path or compound path that connects the two variables. A chain may reverse directions from backward to forward, if necessary, but only one reversal is permitted. It may also pass through a covariance between two exogenous variables, but only one covariance is allowed in each chain. Each path in a chain may be traversed only once. Each chain has an origin or source, which is the "earliest" link in a chain. If a chain has a reversal of direction, the variable at which the reversal occurs is the origin. If there is a covariance between two exogenous variables in the chain, that covariance is the origin (it is the point at which the reversal occurs).

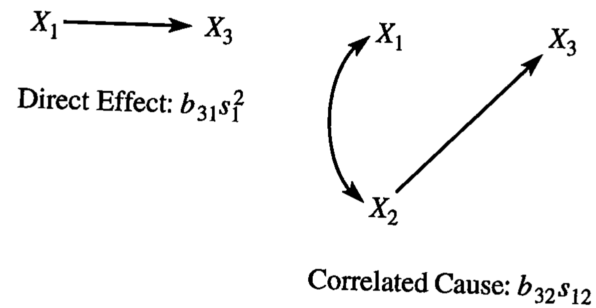
FIGURE 9.1 A Two-Equation Model for the Sample Observations



If there is no reversal, X_i is the origin (it is the earliest point in the chain). To find out how much each chain contributes to the covariance, form the product of all the coefficients along the chain, including either the variance of the origin variable or the covariance at the origin. Finally, the covariance between X_i and X_j equals the sum of the products obtained for all of the distinct chains linking X_i and X_j.

Now, let us apply the above rule to compute the covariances for the variables in the first equation specified by Figure 9.1, the equation for X_3. The chains and the products for each chain that contribute to the covariance between X_3 and X_1 are shown below.

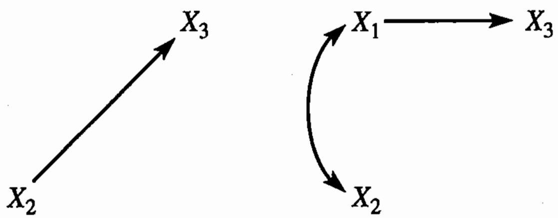
Cov (X_1, X_3)



The first chain is formed by the direct effect of X_1 on X_3. X_1 is the origin of this chain, and its contribution equals the product of the variance of X_1 and the estimated structural coefficient. This contribution is analogous to that derived in Equation 9.3. The second chain is formed by the covariance between X_1 and X_2 and the direct effect of X_2 on X_3. The covariance is the origin. The contribution of the second chain is spurious because it is produced by a correlated cause rather than a direct or indirect effect of X_1 on X_3. We are assuming that there is no causal relationship between the two exogenous variables; they are merely correlated, as the diagram specifies.

The covariance between X_2 and X_3 is analogous to that shown above for X_1 and X_3. The covariance s_{23} is due to a nonspurious direct effect and a spurious correlated cause.

Cov (X_2, X_3)

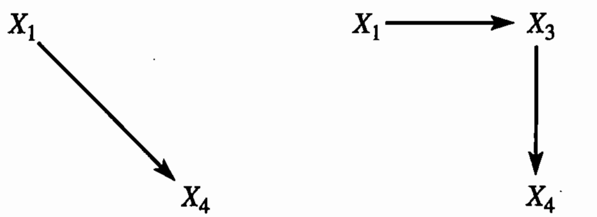


Direct Effect: $b_{32}s_2^2$ **Correlated Cause: $b_{31}s_{12}$**

For both of the above covariances, notice that if the spurious component has the same sign as the causal component (e.g., the direct effects are both positive, and the covariance between the exogenous variables is also positive), we will have redundancy, and the covariance will be inflated. If the spurious component is of opposite sign, however, we will have suppression, and the covariance will be deflated or of opposite sign to that component created by the causal relationship.

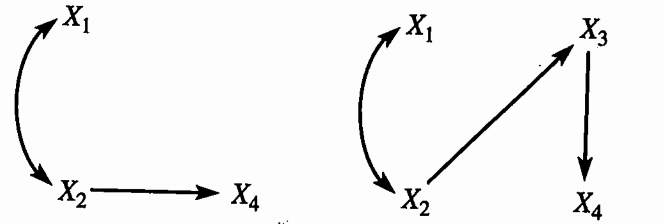
Next, we will use the chain rule to determine the covariances between X_4 and the variables in its equation. The components of the covariance between X_4 and X_1 are more complex because there are three variables that are antecedent to X_4. There are now two causal sources of covariance and two spurious sources.

Cov (X_1, X_4)



Direct Effect: $b_{41}s_1^2$ **Indirect Effect: $b_{43}b_{31}s_1^2$**

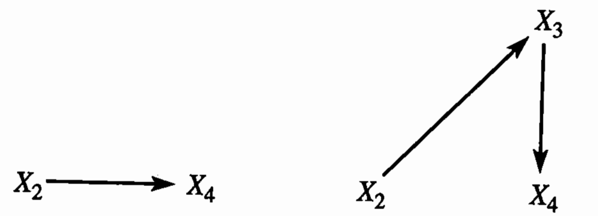
Cov (X_1, X_4) (continued)



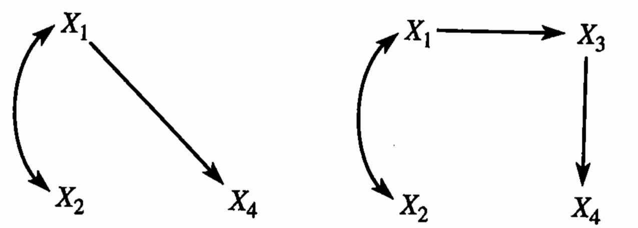
Correlated Cause: $b_{42}s_{12}$ **Correlated Cause: $b_{43}b_{32}s_{12}$**

The chains for the covariance for X_2 and X_4 are shown below. The sources of the covariance between X_2 and X_4 are completely analogous to those for X_1 and X_4: two causal components and two spurious components.

Cov (X_2, X_4)

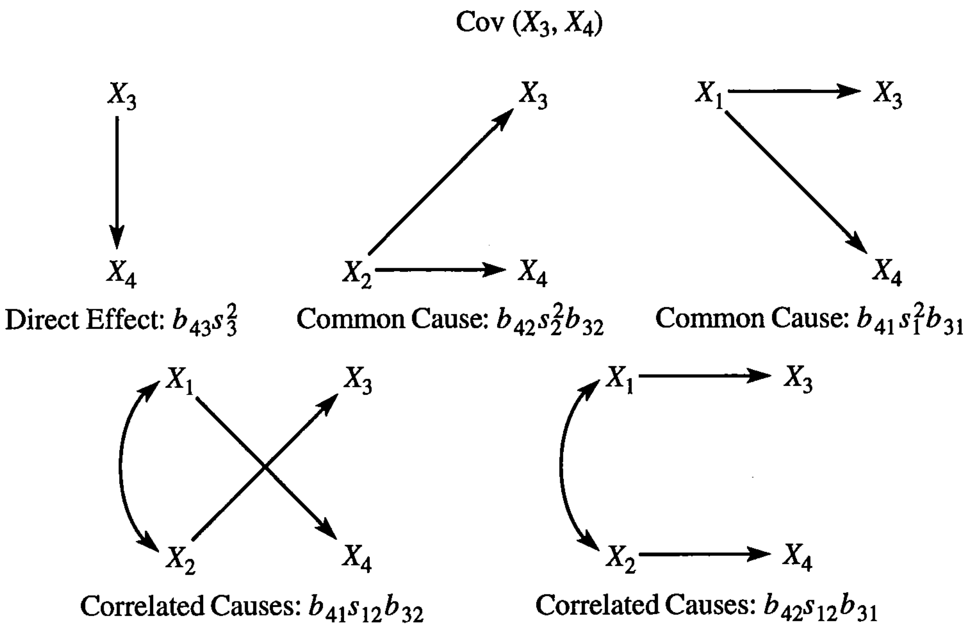


Direct Effect: $b_{42}s_2^2$ **Indirect Effect: $b_{43}b_{32}s_2^2$**



Correlated Cause: $b_{41}s_{12}$ **Correlated Cause: $b_{43}b_{31}s_{12}$**

The covariance between the two endogenous variables, X_3 and X_4, is obtained from the five chains shown below. Since X_3 is dependent upon X_1 and X_2, the possibilities for spurious covariance between X_3 and X_4 are enhanced. There is only one nonspurious source, the direct effect of X_3 on X_4. Depending upon the signs of the various structural coefficients and the covariance between the exogenous variables, the covariance might be either enhanced or reduced due to the various sources of spuriousness.



Each of the five covariances whose components have been diagrammed above may be obtained by summing the products for all of the chains that contribute to the covariance. The covariance between X_1 and X_2 does not have to be computed; it is a given in Figure 9.1.

$$s_{12} = s_{12}$$

$$s_{13} = b_{31}s_1^2 + b_{32}s_{12} \quad \textbf{(9.4)}$$

$$s_{23} = b_{32}s_2^2 + b_{31}s_{12} \quad \textbf{(9.5)}$$

$$s_{14} = b_{41}s_1^2 + b_{43}b_{31}s_1^2 + b_{42}s_{12} + b_{43}b_{32}s_{12} \quad \textbf{(9.6)}$$

$$s_{24} = b_{42}s_2^2 + b_{43}b_{32}s_2^2 + b_{41}s_{12} + b_{43}b_{31}s_{12} \quad \textbf{(9.7)}$$

$$s_{34} = b_{43}s_3^2 + b_{42}s_2^2b_{32} + b_{41}s_1^2b_{31} + b_{42}s_{12}b_{31} + b_{41}s_{12}b_{32} \quad \textbf{(9.8)}$$

The terms in Equations 9.4 through 9.8 consist of the estimated structural coefficients of the model (the b_{ji}'s) and the variances and covariance of the exogenous variables, with one exception. The single exception is in the last equation, which contains the variance of an endogenous variable, X_3. Since X_3 is dependent upon the exogenous variables, however, its variance is created by the exogenous variables. Once we have learned how to write an equation for the variance of X_3, we can enter it into the last equation above to obtain an expression containing only the variances and covariances of the exogenous variables and the structural coefficients.

TABLE 9.1 Variance-Covariance Matrix

	X_1 *EDUC*	X_2 *INCOME*	X_3 *SHORTINC*	X_4 *ESTEEM*
EDUC	7.4475	2.0411	1.1248	.9897
INCOME	2.0411	4.2322	1.3428	.8312
SHORTINC	1.1248	1.3428	1.7767	.7823
ESTEEM	.9897	.8312	.7823	4.8185

TABLE 9.2 Decomposition of Covariances

s_{13}:	$b_{31}s_1^2 = (.0738)(7.4475) =$	.5496
	$b_{32}s_{12} = (.2817)(2.0411) =$	.5750
	$s_{13} =$	1.1246
s_{23}:	$b_{32}s_2^2 = (.2817)(4.2322) =$	1.1922
	$b_{31}s_{12} = (.0738)(2.0411) =$	.1506
	$s_{23} =$	1.3428
s_{14}:	$b_{41}s_1^2 = (.0644)(7.4475) =$	.4796
	$b_{43}b_{31}s_1^2 = (.3612)(.0738)(7.4475) =$	.1985
	$b_{42}s_{12} = (.0507)(2.0411) =$	.1035
	$b_{43}b_{32}s_{12} = (.3612)(.2817)(2.0411) =$	.2077
	$s_{14} =$	.9893
s_{24}:	$b_{42}s_2^2 = (.0507)(4.2322) =$	.2146
	$b_{43}b_{32}s_2^2 = (.3612)(.2817)(4.2322) =$	.4306
	$b_{41}s_{12} = (.0644)(2.0411) =$	.1314
	$b_{43}b_{31}s_{12} = (.3612)(.0738)(2.0411) =$	.0544
	$s_{24} =$	.8310
s_{34}:	$b_{43}s_3^2 = (.3612)(1.7767) =$	.6417
	$b_{42}s_2^2b_{32} = (.0507)(4.2322)(.2817) =$	.0604
	$b_{41}s_1^2b_{31} = (.0644)(7.4475)(.0738) =$	.0354
	$b_{42}s_{12}b_{31} = (.0507)(2.0411)(.0738) =$	.0076
	$b_{41}s_{12}b_{32} = (.0644)(2.0411)(.2817) =$	.0370
	$s_{34} =$	.7821

Covariances for Self-Esteem Model. The self-esteem model (Figure 8.10) and data described in Chapter 8 will be used to illustrate the computation of the components of covariance. The covariances and variances for the four variables in the self-esteem model are shown in Table 9.1.

Equations 9.4 through 9.8 are used to compute the components of the five covariances shown in Table 9.2. Whereas about half of the covariance between

X_1 (EDUC) and X_3 (SHORTINC) is spuriously created by a correlated cause ($b_{32}s_{12}$ = .5750), very little of the covariance between X_2 (INCOME) and X_3 is spurious. With respect to X_1 and X_4 (ESTEEM), however, the covariance that is due to the direct and indirect effects of education (.4796 + .1985 = .6781) is about twice as much as the spurious component created by the correlation of education with income (.1035 + .2077 = .3112). The covariance between income and self-esteem has an even greater nonspurious component (.2146 + .4306 = .6452), relative to its spurious component (.1314 + .0544 = .1850). Finally, even though there are four spurious components to the covariance between X_3 (SHORTINC) and X_4, in comparison to only one causal component ($b_{43}s_3^2$), the great majority of the covariance is nonspurious (.6417/.7821 = .82).

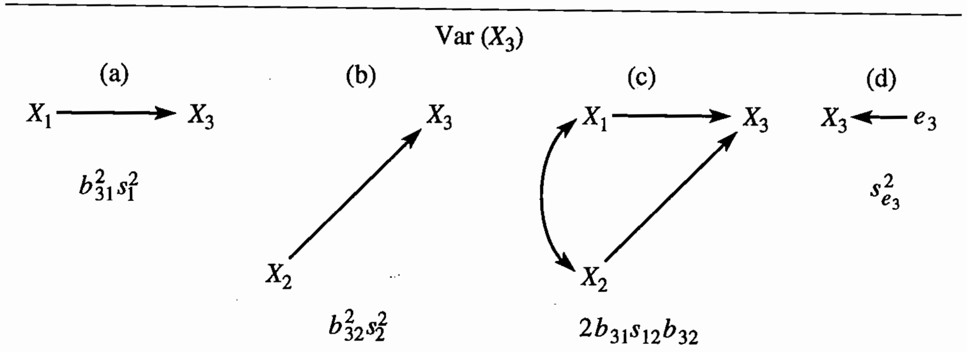
Correlations. Before turning to the equations for the variances of the endogenous variables, we should note what the covariance equations would look like for the path coefficients of a standardized model. The variances of all variables in a standardized model equal unity; thus, the variances in the above equations can all be omitted. Also, the covariance between two standardized variables equals their correlation; thus, the covariances can all be replaced with correlations. Using B_{ji} for a standardized regression coefficient, the equation for the covariance between X_3 and X_4, for example, becomes

$$r_{43} = B_{43} + B_{42}B_{32} + B_{41}B_{31} + B_{42}r_{12}B_{31} + B_{41}r_{12}B_{32}$$

Variances

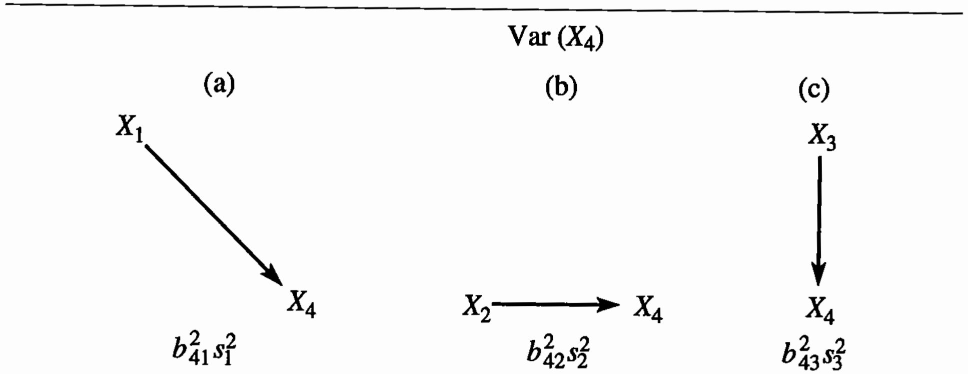
Chain Rule for Variances. The rules for reading the variance of an endogenous variable from a causal diagram are analogous to those for covariances. The variance of a variable can be thought of as the covariance of a variable with itself (see Chapter 2). Therefore, the chains that define contributions to the variance of X_j have X_j at both ends of the chain; the chain starts and ends with X_j. To find each distinct chain, trace backward from X_j to an origin, as defined by the covariance rules, and then trace forward along another path or compound path to get back to X_j. Thus, the chain may be thought of as a loop (but not a causal loop, as in a nonrecursive model) that returns to X_j. There is one difference, however, between a variance chain and a covariance chain. A variance chain passes over the same path twice when the origin is a variable that directly affects X_j; we trace back to a variable that directly affects X_j and then move back to X_j over the same path. The value of a chain's contribution to the variance of X_j is obtained by taking the product of all the coefficients along the chain and the variance, or covariance, at the origin. For chains that do not consist of a direct path between X_j and an antecedent variable, the product must also be multiplied by two, a rule that did not apply to covariance chains.

The rules will be illustrated by reading the variances of X_3 and X_4 from Figure 9.1. The variance for X_3 is shown below.

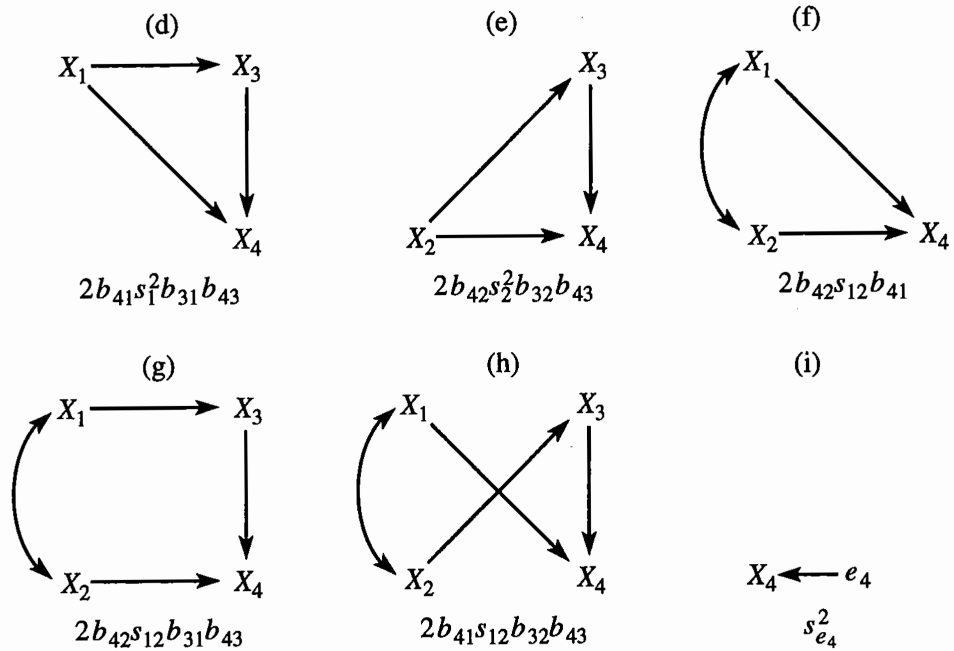


The contributions due to the direct effects of the exogenous variables, (a) and (b), are identical in form to Equation 9.2. Since the variance chain rule indicates that we must trace over the direct path twice, the structural coefficient is squared in (a) and (b). Chain (c) is a new variance component that arises when there are two or more independent variables. Since this component contains the effects of both exogenous variables and their covariance, (c) is a portion of the variance that cannot be allocated to either variable. Since (a) and (b) must be positive because of squaring, the sign of (c) determines whether the covariance between the exogenous variables increases or decreases the variance in X_3. If the direct effects are both positive and there is a positive covariance, for example, the variance in X_3 will be greater than the sum of the direct contributions. If both direct effects are positive but the covariance is negative, the variance will be less than the sum of the direct contributions. If the two exogenous variables are not correlated, however, the variance will equal the sum of the two direct contributions. Thus, correlated input variables may either increase or decrease the diversity/variance of outcome variables relative to inputs that are not correlated. Finally, the contribution of the error term equals its variance since its structural coefficient equals unity by definition.

The variance chains and components for X_4 are as follows:



Var (X_4) (continued)



Chains (a), (b), and (c) define direct contributions analogous to those given before. Chains (d) and (e) each consist of the products of direct and indirect effects of the exogenous variable, times its variance. For example, (d) includes the product of the direct effect of X_1 (b_{41}), the indirect effect of X_1 ($b_{31}b_{43}$), and the variance of X_1. The contribution due to chain (f) is identical in form to chain (c) for the variance of X_3; it involves the direct effects of the exogenous variables and the covariance between them. Chains (g) and (h) also include this covariance; each of these cases, however, includes the product of the direct effect of one exogenous variable and the indirect effect of the other.

Summing the values of each distinct variance chain for X_3 and each distinct chain for X_4 gives the following variance equations:

$$s_3^2 = b_{31}^2 s_1^2 + b_{32}^2 s_2^2 + 2b_{31}s_{12}b_{32} + s_{e_3}^2 \qquad \textbf{(9.9)}$$

$$s_4^2 = b_{41}^2 s_1^2 + b_{42}^2 s_2^2 + b_{43}^2 s_3^2 + 2b_{41}s_1^2 b_{31}b_{43} + 2b_{42}s_2^2 b_{32}b_{43}$$
$$+ 2b_{41}s_{12}b_{42} + 2b_{43}b_{31}s_{12}b_{42} + 2b_{43}b_{32}s_{12}b_{41} + s_{e_4}^2 \qquad \textbf{(9.10)}$$

Equation 9.10 is rather lengthy. It demonstrates that the variance of a dependent variable can be a rather complex function of the direct effects, indirect effects, variances, and covariances of the independent variables, even in a relatively simple model containing only three independent variables. Some of these terms, however, might be zero or nearly zero when empirically estimated; thus, the equation might turn out to be less complicated than it appears.

It is tempting to try to use Equations 9.9 and 9.10 to divide the explained variance into components that can be allocated to each independent variable and to divide the variance that each variable explains into various direct and indirect components. Notice that each term in each equation contains either a variance of an independent variable X_i or a covariance between two exogenous variables. We might sum the terms containing s_i^2 and allocate this portion of the explained variance to X_i. The desire to allocate the explained variance among the independent variables runs into problems, however, because of the terms that contain covariances. These terms represent amounts by which variance is *amplified* or *dampened* due to the correlated causes. Therefore, these portions cannot be allocated to a specific variable. If the covariance is small, or if one of the variables has a small effect, then the value of the term containing the covariance might be quite small and of no importance. If most of the terms containing covariances are small, then most of the variance may be allocated among the various independent variables. If there is a great deal of multicollinearity, however, then large portions of the explained variance will go unallocated. Thus, the degree to which Equations 9.9 and 9.10 can be used to allocate variance among the independent variables will have to be determined on a case-by-case basis.

There is also a problem in dividing explained variance into direct and indirect components: some of the terms containing s_i^2 also contain products of the direct and indirect effects of X_i, such as $2b_{41}s_1^2b_{31}b_{43}$. The explained variance represented by this term cannot be allocated to either the direct effect or the indirect effect. Notice that this term may be either positive or negative; if the direct effect (b_{41}) and the indirect effect ($b_{31}b_{43}$) have the same signs, the term will be positive, and if they have opposite signs, it will be negative. If the term is positive, the variance explained by direct and indirect effects combined will be greater than would be expected by summing the variances that each would explain in the absence of the other. If they have opposite signs, the combined explained variance will be less than the sum of the variances that each would explain in the absence of the other. Thus, the terms that contain the products of direct and indirect effects prevent us from dividing the variance explained by an independent variable into direct and indirect components.

Variances for Self-Esteem Model. Equations 9.9 and 9.10 are used to decompose the variances of X_3 (SHORTINC) and X_4 (ESTEEM) in Table 9.3. Over 70 percent of the explained variance in X_4 is allocated to the direct effect of INCOME (.3358/.4613 = .728). The positive covariance between the two exogenous variables X_1 and X_2, however, does increase the variance in SHORTINC some-

TABLE 9.3 Decomposition of Variances

s_3^2:	$b_{31}^2 s_1^2 = (.0738)^2(7.4475) =$	.0406
	$b_{32}^2 s_2^2 = (.2817)^2(4.2322) =$	.3358
	$2b_{31}s_{12}b_{32} = 2(.0738)(2.0411)(.2817) =$	.0849
	$s_{e_3}^2 = (1 - R_3^2)s_3^2 = (1 - .2596)(1.7767) =$	1.3155
	$s_3^2 =$	1.7768
s_4^2:	$b_{41}^2 s_1^2 = (.0644)^2(7.4475) =$	.0309
	$b_{42}^2 s_2^2 = (.0507)^2(4.2322) =$	.0109
	$b_{43}^2 s_3^2 = (.3612)^2(1.7767) =$	.2318
	$2b_{41}s_1^2 b_{31}b_{43} = 2(.0644)(7.4475)(.0738)(.3612) =$	.0256
	$2b_{42}s_2^2 b_{32}b_{43} = 2(.0507)(4.2322)(.2817)(.3612) =$	.0437
	$2b_{41}s_{12}b_{42} = 2(.0544)(2.0411)(.0507) =$	.0133
	$2b_{43}b_{31}s_{12}b_{42} = 2(.3612)(.0738)(2.0411)(.0644) =$	.0055
	$2b_{43}b_{32}s_{12}b_{41} = 2(.3612)(.2817)(2.0411)(.0644) =$	.0267
	$s_{e_4}^2 = (1 - R_4^2)s_4^2 = (1 - .0806)(4.8185) =$	4.4301
	$s_4^2 =$	4.8185

what (.0849). With respect to ESTEEM, the majority of the explained variance is allocated to the direct effect of SHORTINC [.2318/(4.8185 – 4.4301) = .60]. Even though there are three components containing s_{12}, the positive covariance between the two exogenous variables does very little to increase the variance in self-esteem (.0133 + .0055 + .0267 = .0455).

Reduced-Form Variances and Covariances

It is also important to note that Equation 9.10 also contains a term that includes the variance of X_3, namely, $b_{43}^2 s_3^2$. This represents variance in X_4 that is directly caused by X_3, which is an endogenous variable. There is nothing wrong with including a term that represents the variance in one endogenous variable that is explained by another endogenous variable. However, the variance of X_3 is partly caused by the two exogenous variables. Thus, this term represents variance in X_4 that is partly caused by the indirect effects of X_1 and X_2 that pass through X_3. If we substitute Equation 9.9 (the equation for the variance of X_3) for s_3^2 in Equation 9.10, we can derive an equation that contains terms for all of the indirect effects of the exogenous variables:

$$s_4^2 = b_{41}^2 s_1^2 + b_{42}^2 s_2^2 + b_{43}^2(b_{31}^2 s_1^2 + b_{32}^2 s_2^2 + 2b_{31}s_{12}b_{32} + s_{e_3}^2) + 2b_{41}s_1^2 b_{31}b_{43}$$
$$+ 2b_{42}s_2^2 b_{32}b_{43} + 2b_{41}s_{12}b_{42} + 2b_{43}b_{31}s_{12}b_{42} + 2b_{43}b_{32}s_{12}b_{41} + s_{e_4}^2$$

The expression in parentheses is Equation 9.9. If we multiply the parenthetical expression by b_{43}^2 and rearrange slightly, we get

$$\begin{aligned} s_4^2 = {} & b_{41}^2 s_1^2 + b_{42}^2 s_2^2 \\ & + b_{43}^2 b_{31}^2 s_1^2 + b_{43}^2 b_{32}^2 s_2^2 \\ & + 2b_{41}s_1^2 b_{31}b_{43} + 2b_{42}s_2^2 b_{32}b_{43} \\ & + 2b_{41}s_{12}b_{42} + 2b_{43}b_{31}s_{12}b_{42} + 2b_{43}b_{32}s_{12}b_{41} + 2b_{43}b_{31}s_{12}b_{32}b_{43} \\ & + b_{43}^2 s_{e_3}^2 + s_{e_4}^2 \end{aligned} \quad \textbf{(9.11)}$$

At first glance, Equation 9.11 appears to be quite imposing. However, let us note the structure in the equation. First, Equation 9.11 is a *reduced-form equation* because X_3 has been eliminated. As such, each term contains either a variance of an exogenous variable or a covariance between two exogenous variables, with the error terms rightfully defined as exogenous variables. Therefore, Equation 9.11 shows that the variance of an endogenous variable can be accounted for entirely by the structural coefficients of the model plus the variances and covariances of the exogenous variables. The first row contains terms for the variance caused by direct effects. The second row contains terms for the variance caused by indirect effects. The terms in the third row include the products of the direct and indirect effects for each exogenous variable. Thus, Row 3 shows that the variance cannot be divided between direct and indirect effects. The fourth row contains correlated effects (both direct and indirect) between variables. Row 4 shows that the explained variance cannot be divided up between the exogenous variables when they are correlated. Row 5 contains the variance contributed by the error terms.

Note the next-to-last term in the equation, $b_{43}^2 s_{e_3}^2$. This is the variance contributed by the error term for X_3. It is the variance in X_4 caused by variance in X_3 that is not explained by X_1 and X_2. Thus, it is the unique contribution that X_3 makes to the variance of X_4. It is *analogous to a squared semipartial correlation*, except that it is the amount of variance, rather than the proportion of variance, accounted for by X_3.

Equation 9.11 can be simplified considerably by writing it in terms of total effects (T_{ji}'s) instead of structural coefficients (b_{ji}'s), as follows:

$$\begin{aligned} s_4^2 &= (b_{41} + b_{31}b_{43})^2 s_1^2 + (b_{42} + b_{32}b_{43})^2 s_2^2 \\ &\quad + 2(b_{41} + b_{31}b_{43})s_{12}(b_{42} + b_{32}b_{43}) + b_{43}^2 s_{e_3}^2 + s_{e_4}^2 \\ &= T_{41}^2 s_1^2 + T_{42}^2 s_2^2 + 2T_{41}s_{12}T_{42} + T_{43}^2 s_{e_3}^2 + s_{e_4}^2 \end{aligned} \quad \textbf{(9.12)}$$

The terms $(b_{41} + b_{31}b_{43})$ and $(b_{42} + b_{32}b_{43})$ in the first row of Equation 9.12 are the total effects of X_1 and X_2, respectively. Equation 9.12 expresses the variance of X_4 in terms of the variance of each exogenous variable times its squared total effect (the total effect of e_4 equals unity), with one exception. The presence of the term $2T_{41}s_{12}T_{42}$ shows that part of the variance of X_4 is due to the correlated total effects of X_1 and X_2. Thus, there is still a component of the variance of X_4 that cannot be attributed uniquely to X_1 or X_2.

Now we return to Equation 9.8 for the covariance between X_3 and X_4, an equation that also contained the variance of the endogenous variable X_3. If we substitute Equation 9.9 for s_3^2 in Equation 9.8, we get

$$s_{34} = b_{41}s_1^2b_{31} + b_{42}s_2^2b_{32} + b_{31}s_{12}b_{42} + b_{32}s_{12}b_{41} + b_{43}b_{31}^2s_1^2 + b_{43}b_{32}^2s_2^2 + 2b_{43}b_{31}s_{12}b_{32} + b_{43}s_{e_3}^2 \qquad \textbf{(9.13)}$$

This is a reduced-form equation containing only variances and covariances of exogenous variables. The terms in the second row that contain the covariance between the exogenous variables show that we cannot allocate the covariance of X_3 and X_4 to unique contributions made by X_1 and X_2, due to the covariance between the latter two variables.

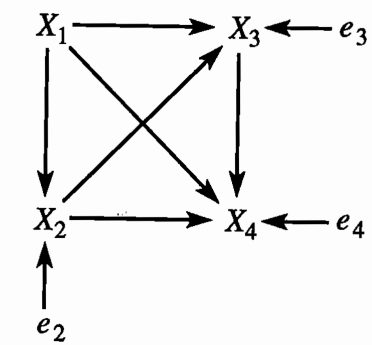
Self-Esteem Example. Equation 9.12 is used to allocate the variance of self-esteem (X_4) among the exogenous variables (Table 9.4). Education and income are now allocated more variance (.0617 + .0984 + .0568 = .2169) than is SHORTINC ($T_{43}^2s_{e_3}^2$ = .1716). This is because the variance in self-esteem caused by the variance in SHORTINC that is explained by education and income is now allocated to education and income. Only the variance in self-esteem caused by the unique variance in SHORTINC ($s_{e_3}^2$) is allocated to SHORTINC.

Equation 9.13 is used to allocate the covariance between X_3 and X_4 among the exogenous variables (Table 9.4). In Table 9.2, eighty-two percent of the covariance was attributed to the direct effect of X_3. The reduced-form equation now allocates some of that component to X_1 and X_2 (.0147 + .1213 + .0307 = .1667). Most of that component, however, is attributed to the unique variance of X_3 ($b_{43}s_{e_3}^2$ = .4751).

TABLE 9.4 Reduced-Form Components of Variance and Covariance

s_4^2:	$T_{41}^2s_1^2 = (.0911)^2(7.4975) =$	.0617
	$T_{42}^2s_2^2 = (.1525)^2(4.2322) =$	.0984
	$2T_{41}^2s_{12}T_{42} = 2(.0911)(2.0411)(.1525) =$	.0568
	$T_{43}^2s_{e_3}^2 = (.3612)^2(1.3154) =$	.1716
	$s_{e_4}^2 = (1 - R_4^2)s_4^2 = (1 - .0806)(4.8185) =$	4.4301
	$s_4^2 =$	4.8186
s_{43}:	$b_{42}^2s_2^2b_{32} = (.0507)(4.2322)(.2817) =$	.0604
	$b_{41}s_1^2b_{31} = (.0644)(7.4475)(.0738) =$	.0354
	$b_{42}s_{12}b_{31} = (.0507)(2.0411)(.0738) =$	.0076
	$b_{41}s_{12}b_{32} = (.0644)(2.0411)(.2817) =$	.0370
	$b_{43}b_{31}^2s_1^2 = (.3612)(.0738)^2(7.4475) =$	.0147
	$b_{43}b_{32}^2s_2^2 = (.3612)(.2817)^2(4.2322) =$	.1213
	$2b_{43}b_{31}s_{12}b_{32} = 2(.3612)(.0738)(2.0411)(.2817) =$	.0307
	$b_{43}s_{e_3}^2 = (.3612)(1.3154)$	.4751
	$s_{43} =$	.7821

FIGURE 9.2 Three-Equation Model



A Three-Equation Model

Covariances

We will now modify Figure 9.1 by specifying a causal path from X_1 to X_2 (Figure 9.2). We now have an equation for X_2. We will use the chain rules for reading covariances from the diagram. All of the chains that previously included a covariance between X_1 and X_2 will change; the other chains will remain the same. The covariance between X_1 and X_2 is now read as

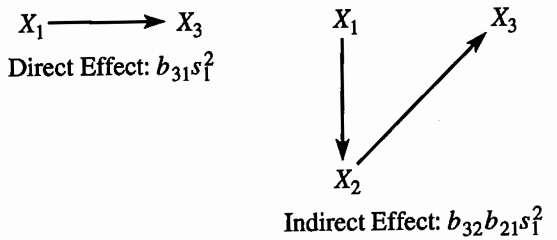
Cov(X_1, X_2)

Direct Effect: $b_{21}s_1^2$

The fact that this covariance can now be expressed as a structural coefficient times the variance of the exogenous variable will have major ramifications for the remainder of the covariances.

The covariance between X_1 and X_3 is

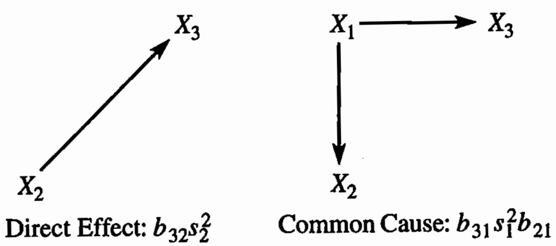
$\mathrm{Cov}(X_1, X_3)$



The second chain was a correlated effect in the two-equation model, but now it is an indirect-effect contribution to the covariance; it is no longer a spurious component of covariance. The change in the expression and its interpretation is due to the newly specified causal path between X_1 and X_2.

The covariance between X_2 and X_3 is given by

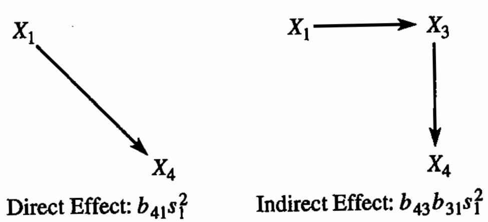
$\mathrm{Cov}(X_2, X_3)$

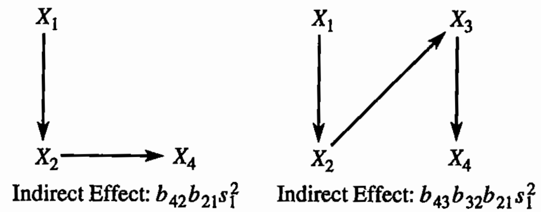


The second chain was previously due to a correlated cause but now it is due to a common cause. In this case, the change in specification does not alter the conclusion that the second component is a spurious contribution to the covariance.

The covariance between X_1 and X_4 is read as follows:

$\mathrm{Cov}\ (X_1, X_4)$

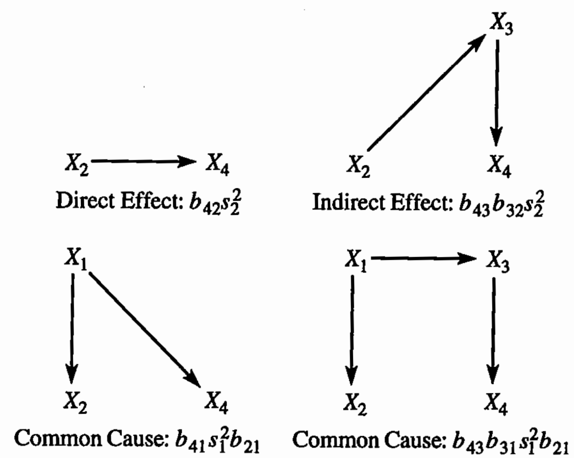




Whereas in the two-equation model the bottom two chains were spurious contributions, in the three-equation model we are using now, they are due to indirect effects and thus are nonspurious. All of the covariance between these two variables is now a valid causal relationship (nonspurious).

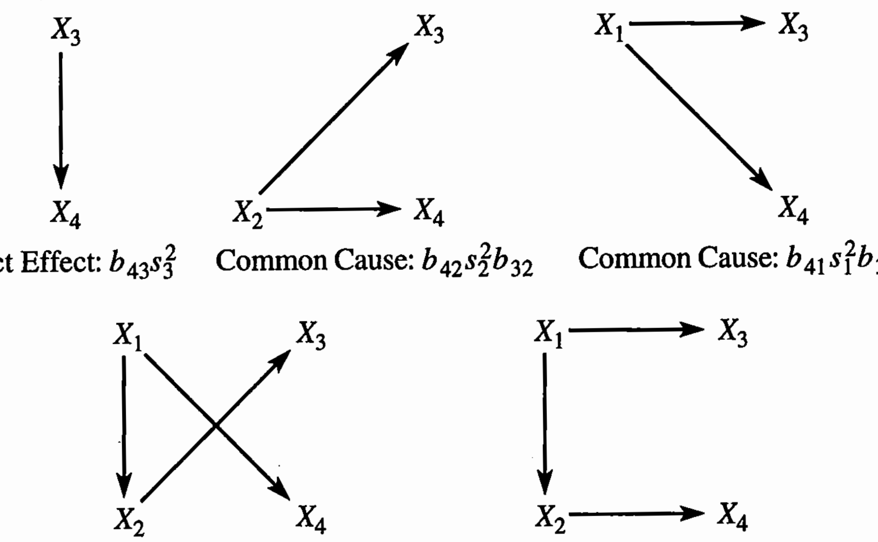
The covariance between X_2 and X_4 is diagrammed next. The bottom two chains were previously read as correlated causes and thus were spurious; they are now read as common causes, but they are still spurious.

$\mathrm{Cov}\ (X_2, X_4)$



The covariance between X_3 and X_4 is shown below. The bottom two chains were due to correlated causes in the two-equation model. They are now due to a common cause, since X_1 is now the origin of the chain. They both are still spurious sources of variance. All of the sources are spurious except for the covariance created by the direct effect.

Cov (X_3, X_4)



Direct Effect: $b_{43}s_3^2$ Common Cause: $b_{42}s_2^2b_{32}$ Common Cause: $b_{41}s_1^2b_{31}$

Common Cause: $b_{41}s_1^2b_{21}b_{32}$ Common Cause: $b_{42}b_{21}s_1^2b_{31}$

Summing the products obtained for each distinct covariance chain gives the following covariance equations:

$$s_{12} = b_{21}s_1^2$$
$$s_{13} = b_{31}s_1^2 + b_{32}b_{21}s_1^2$$
$$s_{23} = b_{32}s_2^2 + b_{31}s_1^2b_{21}$$
$$s_{14} = b_{41}s_1^2 + b_{43}b_{31}s_1^2 + b_{42}b_{21}s_1^2 + b_{43}b_{32}b_{21}s_1^2$$
$$s_{24} = b_{42}s_2^2 + b_{43}b_{32}s_2^2 + b_{41}s_1^2b_{21} + b_{43}b_{31}s_1^2b_{21}$$
$$s_{34} = b_{43}s_3^2 + b_{42}s_2^2b_{32} + b_{41}s_1^2b_{31} + b_{31}s_1^2b_{21}b_{42} + b_{32}b_{21}s_1^2b_{41}$$

Each term in the equation for each s_{ij} includes a variance. If that variance is s_i^2, then that term represents a nonspurious source of covariance. If the variance is not s_i^2, then the term represents a spurious source of covariance. There are no correlated exogenous variables in the three-equation model, since there is only one measured exogenous variable in the model. Consequently, there are no covariances in the above equations. Where s_{12} appeared in Equations 9.4 through 9.8, it has been replaced by $b_{21}s_1^2$ in the above equations. Therefore, each term in the above equations will have the same value it had in Equations

9.4 and 9.5. For example, the values of the covariance components of the three-equation self-esteem model are the same as those given in Table 9.2. Since a variance appears in each term of the above equations, each chain's contribution to the covariance s_{ij} can be assigned to a specific variable. This was not always possible with the two-equation model because that model did not specify a causal relationship between X_1 and X_2.

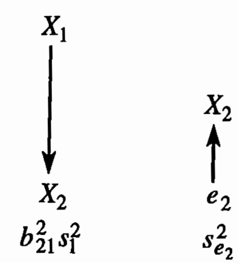
The decomposition of covariances is similar to the decomposition of the bivariate slope that was covered earlier. The sum of the nonspurious covariance components is analogous to the total effect of an independent variable on a dependent variable. The sum of the spurious covariance components is analogous to the difference between the bivariate slope and the total effect.

Some of the above covariance equations contain variances of endogenous variables. After reading equations for the variances of the endogenous variables from the diagram of the three-equation model (Figure 9.2), we could substitute them in the above equations to obtain equations containing only variances of the exogenous variables. These reduced-form equations would be analogous to Equation 9.13 for the covariance between X_3 and X_4. It is recommended that you derive these equations in order to demonstrate that all of the covariances can be accounted for by the structural coefficients and the variances of the exogenous variables.

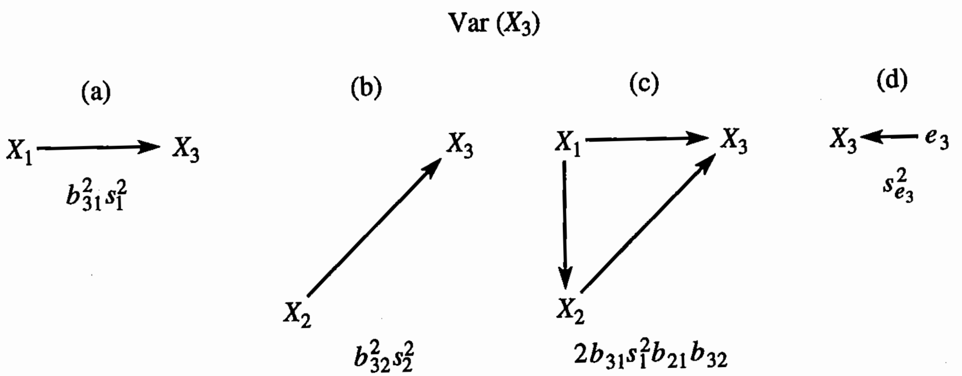
Variances

We will now read the equations for the variances of the endogenous variables from Figure 9.2. The variance of X_2 was not accounted for by the two-equation model.

Var (X_2)

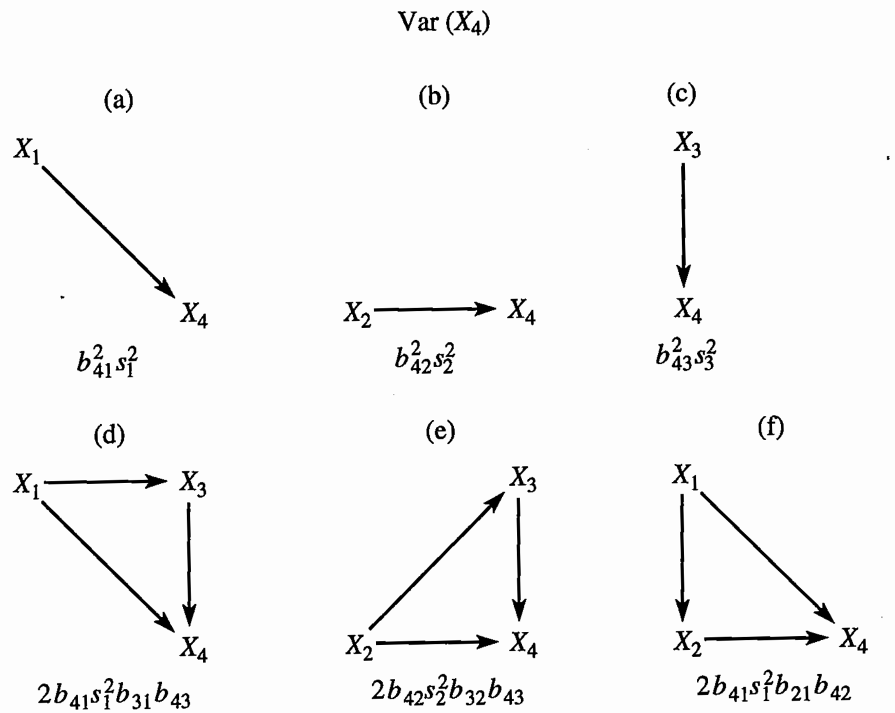


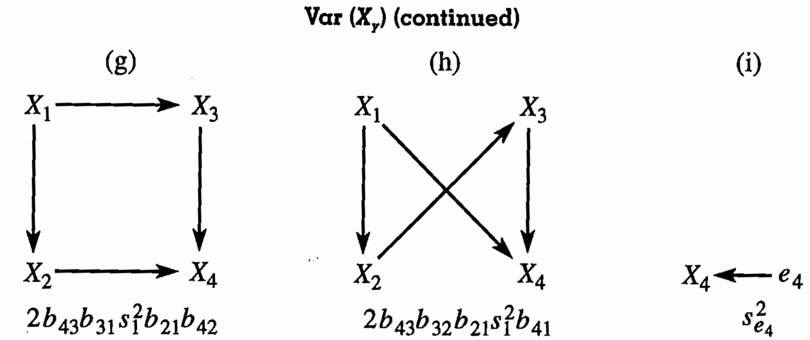
The variance of X_3 is given by



Component (c) is the only one that changes in the three-equation model. It was previously a portion of variance due to correlated causes. Now (c) is attributed to X_1.

The sources of variance in X_4 are shown below.





Components (f), (g), and (h) are the three components that have changed from the first model. They were sources of variance due to correlated causes in the first model.

The equations for the variances of the endogenous variables are

$$s_2^2 = b_{21}^2 s_1^2 + s_{e_2}^2 \quad \textbf{(9.14)}$$

$$s_3^2 = b_{31}^2 s_1^2 + b_{32}^2 s_2^2 + 2b_{31} s_1^2 b_{21} b_{32} + s_{e_3}^2 \quad \textbf{(9.15)}$$

$$s_4^2 = b_{41}^2 s_1^2 + b_{42}^2 s_2^2 + b_{43}^2 s_3^2 + 2b_{41} s_1^2 b_{31} b_{43} + 2b_{42} s_2^2 b_{32} b_{43} + 2b_{41} s_1^2 b_{21} b_{42} + 2b_{43} b_{31} s_1^2 b_{21} b_{42} + 2b_{43} b_{32} b_{21} s_1^2 b_{41} + s_{e_4}^2 \quad \textbf{(9.16)}$$

Equation 9.14 is new for this model. Each term in Equations 9.15 and 9.16 contains a variance of one of the variables; there are no longer any covariances in the equations. Where s_{12} appeared in Equations 9.9 and 9.10, it has been replaced by $b_{21} s_1^2$ in Equations 9.15 and 9.16, respectively. Thus, the components of Equations 9.15 and 9.16 would have the same numeric values as those given in Table 9.3 for the self-esteem example.

Some of the variances in Equations 9.15 and 9.16, however, are the variances of the endogenous variables X_2 and X_3. Since s_2^2 and s_3^2 are partly determined by X_1 and by X_1 and X_2, respectively, these terms include some of the variance that is indirectly caused by X_1 and X_2. The only way to achieve an unambiguous allocation of the variance of X_3 and X_4 is to remove the variance of X_2 from Equation 9.15 and to remove the variances of X_2 and X_3 from Equation 9.16. This may be accomplished by substituting Equation 9.14 for s_2^2 in Equation 9.15 and by substituting Equations 9.14 and 9.15 for s_2^2 and s_3^2, respectively, in Equation 9.16. The result of this tedious operation, after much rearranging and simplifying, is

$$s_3^2 = (b_{31} + b_{21} b_{32})^2 s_1^2 + b_{32}^2 s_{e_2}^2 + s_{e_3}^2 \quad \textbf{(9.17)}$$

$$s_4^2 = (b_{41} + b_{21} b_{42} + b_{31} b_{43} + b_{21} b_{32} b_{43})^2 s_1^2 + (b_{42} + b_{32} b_{43})^2 s_{e_2}^2 + b_{43}^2 s_{e_3}^2 + s_{e_4}^2 \quad \textbf{(9.18)}$$

Let us examine carefully the terms in these equations. The terms in parentheses in Equation 9.17 are the direct and indirect effects of X_1 on X_3. The sum of these effects is the total effect of X_1 on X_3. When this total effect is squared and multiplied times the variance of X_1, we get the total variance caused by X_1. The next term in Equation 9.17, $b_{32}^2 s_{e_2}^2$, is the square of the direct effect of X_2 on X_3 multiplied by the error variance of X_2. This error variance is the portion that is not explained by X_1, i.e., the unique variance of X_2 relative to X_1. Thus, the second term in Equation 9.17 is the amount of variance in X_3 that is uniquely caused by X_2 or the total variance explained by X_2. The last term, of course, is the residual variance in X_3 that cannot be explained by X_1 and X_2. Thus, Equation 9.17 equals the total variance explained by X_1, plus the total variance explained by X_2, plus the unexplained variance of X_3. Thus, we have arrived at an unambiguous allocation of the variance among the variables.

Equation 9.18, although more complex, has the same interpretation. The terms in parentheses are the total effects of X_1 and X_2 on X_4. The equation also contains the square of the direct effect of X_3 on X_4, which is the total effect of X_3. When these terms are multiplied by the appropriate variances, we get the total variance explained by each independent variable.

We should note that when we square these expressions that represent the total effects, we will get some terms that consist of the product of direct and indirect effects. These products cannot be allocated to either the direct or the indirect effect. Thus, it still is not possible to say how much of the total explained variance is due to the direct effect and how much is due to each of the indirect effects. We must be content to know the total variance explained by the sum of the direct and indirect effects.

Since the equations contain total effects, we may simplify them as follows:

$$s_3^2 = T_{31}^2 s_1^2 + T_{32}^2 s_{e_2}^2 + s_{e_3}^2$$

$$s_4^2 = T_{41}^2 s_1^2 + T_{42}^2 s_{e_2}^2 + T_{43}^2 s_{e_3}^2 + s_{e_4}^2$$

It is important to remember how we were able to accomplish this unambiguous decomposition of the variance. It was made possible by the fact that we were able to specify a causal order for *all* of the variables in the model. If we were not able to do this for some set of variables being investigated, we would have two or more exogenous variables that are merely correlated but that are not believed to be involved in a causal relationship. In that case, Equations 9.17 and 9.18 would include some terms that contain a covariance between the exogenous variables. This would result in some portion of the variance that could not be allocated to one variable or the other.

Summary

A chain rule for reading covariance equations from a path diagram was presented. Each component of the equation for s_{ij} consists of a chain of variables with X_j at one end and X_i at the other end. The contribution of the chain equals the product of all the structural coefficients along the chain times either the variance of the "earliest" variable or the covariance between two exogenous variables. Each component represents either a nonspurious contribution to the covariance, resulting from either a direct or indirect effect, or a spurious contribution, resulting from a common cause or correlated cause. Elaborating a model by specifying a causal link between some of the exogenous variables will change some of the spurious components to nonspurious indirect effects. The reduced-form version of the covariance equation allocates all of the covariance to the exogenous variables, both measured and unmeasured.

The chain rule for reading variance equations from a path diagram is similar to that for covariances, except that each chain begins and ends with X_j. The variance components in models with two or more exogenous variables consist of contributions due to direct effects, combinations of direct and indirect effects, and correlated causes. It is not possible to allocate the explained variance between direct-effect components and indirect-effect components. It is also not possible to allocate unambiguously the variance of an endogenous variable into components due to each of the independent variables. If, however, the model can be respecified to make endogenous variables out of all but one of the exogenous variables (the three-equation example), then the explained variance can be allocated into distinct components due to each of the independent variables. Finally, the reduced forms of the variance equations allocate the variance of each endogenous variable into unique components that represent the total effect of the single measured exogenous variable and the total effect of each unmeasured exogenous variable.

Reference

Wright, Sewell. 1921. "Correlation and Causation." *Journal of Agricultural Research* 20:557–585.